The Dissertation Committee for Chengcheng Liu
certifies that this is the approved version of the following dissertation:

Stability and Pricing in Naor's Model with Arrival Rate Uncertainty

Committee:

John J. Hasenbein, Supervisor

James Eric Bickel

Grani A. Hanasusanto

Milica Cudina

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by

Chengcheng Liu

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Stability and Pricing in Naor's Model with Arrival Rate Uncertainty

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Supervisor: John J. Hasenbein

Naor's observable queueing model describes an $M/M/1$ queue with strategic customers and a system manager who maximizes the long-run average revenue rate or social benefit rate. Customers have identical service values and waiting time costs, assuming the waiting cost is linear in time. A new customer chooses to either enter the system or balk after observing the queue length. The system manager decides on the admission fee, which is assumed to be a constant. The results of Naor's model are: the optimal policy for customers is a threshold policy, and customers enter if and only if the queue length is no larger than a threshold; the revenue-maximizing threshold is no larger than the socially optimal threshold, or equivalently, a revenue maximizer (RM) charges a fee no less than a social optimizer (SO).

This research studies an observable queueing system in which the arrival rate is not known with certainty by either customers or the system manager. The customer population is modeled to be either homogeneous or heterogeneous. We present three different models: static pricing with uncertain arrival rate and heterogeneous customers; state-dependent pricing with uncertain arrival rate and homogeneous customers; and state-dependent pricing with uncertain arrival rate and heterogeneous customers. We study the system stability, the optimal behavior of customers and the optimal pricing policies of the system manager.

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Chapter 1

Introduction

1.1 Motivation

There are many queueing systems in real life: a line of customers at a restaurant waiting to be seated; a line in front of a movie theater ticket window; or queues of visitors waiting for a ride in the theme park. In such scenarios, any new customer who just arrives may have the following concerns: How long is the queue? How fast the line is moving? How good is the food in this restaurant? How much does a ticket cost? Is there any other option besides waiting in this line? It is interesting to observe that each customer acts independently in a self-interested manner. Customers could have different benefit-cost analyses based on their self-interest. They may decide to join the queue, balk, renege the decision after they join, or revisit the queue multiple times in order to find a better time to join the queue.

In 1969, Naor [15] was the first to publish his work on a strategic queueing model that incorporates the importance of pricing in a queueing system to regulate demand and optimize social benefit or revenues. In almost every service system, customers independently act in order to maximize their benefit. Components of a customer's welfare might include: the service rate

- the average rate of complete service; service value - the value received from the service; waiting time - the time until service is completed; and the fee charged in order to get served. So the optimal behavior of a single customer is also affected by other customers, servers or the system manager. Hence, game theory and equilibrium behavior of customers must now be incorporated into the model. Pricing plays a critical role in such game theoretic considerations, assuming customers are price and delay sensitive.

In terms of pricing a service, there are some examples of interest. One example is the pricing strategy in transportation systems. The I-35E TEXpress Lanes in Texas opened in May 2017. Drivers have the alternatives to take the main lanes for free or pay to use the TEXpress Lanes. The TEXpress Lanes are (in theory) managed to keep traffic moving at 50 mph or faster. Traffic speeds are maintained through dynamic pricing based on road congestion. Lee County in Florida applies variable pricing on the Cape Coral and Midpoint toll bridges and offers specific discount hours. Drivers are encouraged to shift from peak periods to discount periods. New York City has proposed charging vehicles for driving in the congested southern half of Manhattan. This congestion pricing encourages passengers to take public transit and the tolls will be used to support the expected increase in transit ridership.

After Naor's paper, there have been numerous papers published on the optimal control of strategic queues. These papers typically study the behavior of a revenue maximizing firm or a manager interested in maximizing the social benefit. A major subject of research is on the effect of information on decision makers. In this research we are interested in two information levels: the uncertain knowledge of system parameters and the asymmetric information when customers are heterogeneous and the system manager cannot distinguish among customers.

1.2 Naor's Model and Extensions

In Naor's seminal queueing model [15], potential customers arrive according to a Poisson process and receive i.i.d. exponentially distributed services. These customers decide whether or not to enter the queue by observing the queue length and comparing their expected waiting cost to their fixed value for receiving service, entering only if the net expected benefit is non-negative. In this "observable" model customers are allowed to balk, but are not allowed to renege. In Naor's paper, and many others, this framework has been explored with three different control schemes: no manager, a social optimizer (SO) and a revenue maximizer (RM). The SO strives to maximize social benefit rate, usually by imposing an admission fee, whereas the RM wishes to maximize the revenue rate.

In Naor's model [15] of observable queues with a deterministic arrival rate and homogeneous customers, the system manager utilizes a static pricing approach in which the admission fee is fixed. The customers' optimal policy is a threshold policy, i.e., they enter the system only if the current system size is below a fixed value. In our study, and much previous work, it is important to identify properties of the $S(p)$, the social benefit rate as a function of the admission fee p, and $Z(p)$, the revenue rate as a function of p. Naor proved that the revenue-maximizing fee p^*_Z is no less than the social-optimizing fee p^*_S, and thus that the revenue-maximizing threshold n^*_Z is no greater than the social-optimizing threshold n^*_S.

Some models consider customer heterogeneity. Edelson and Hildebrand [7] assume that the waiting cost rate has a discrete distribution with two possible values, c_1 and c_2. They observe that $S(p)$ is unimodal but that $Z(p)$ can be bimodal. Furthermore, they show that the usual relation $p^*_S \leq p^*_Z$ does not necessarily hold. Schroeter [20] modifies Naor's model by assuming that the waiting cost rate is uniformly distributed over an interval $[0, c_{max}]$ and simplifies the derivation of the profit-maximizing price. Larsen [13] gives a numerical experiment by assuming the service value is a continuous uniform random variable, and finds that both $S(p)$ and $Z(p)$ are unimodal in p. He also proves that $p^*_S \leq p^*_Z$ in the special case in which customers enter if and only if the system is empty. Para-Fruos and Aranda-Gallego [17] assume two classes of customers with different time values and observe that $p^*_S \leq p^*_Z$ does not hold. Hassin [9] considers an M/M/1 unobservable queue where the service value or the waiting cost rate is a random variable obtaining one of two given values with known probabilities. For each case, he investigates three subcases: customers are uninformed of the realization of the random variable and the server sets a single price independent of the realization; customers are informed and the the server sets a sing price depending on the realization; customers are informed, but the server sets a single price before the random variable is realized. Related conclusions are that if the service value is uncertain, in terms of the revenue rate, it is better to not inform customers when customers have small waiting cost rate, but worse when they have large waiting cost rate; if the waiting cost rate is uncertain, the social welfare is the same whether to inform customers or not, given that the price induces positive demand. Sun and Li [21] extend [9] by allowing random parameter to obtain more than two values and get similar results.

Some papers investigate state dependent pricing by allowing various fees as the queue length changes. Lipmann and Stidham [14] study a system with a state dependent fee to achieve social optimality, considering holding costs and discounting. Chen and Frank [5] study a system with a state-dependent pricing structure, in which the admission fee depends on the queue length. When customers are homogeneous, customers follow a threshold policy. In this case, the SO and RM have the same optimal pricing policies and the admission fee is decreasing in the queue length. They also provide an extension in which customers have two possible service values. Here, the SO follows the same optimal policy as in the homogeneous case by treating the two types of customers as indistinguishable. However, the RM follows a two-threshold policy in an attempt to extract more revenue from the high service-value customers while possibly sacrificing low-value customers.

The work on arrival rate uncertainty is limited. Besbes and Maglaras [4] examine an M/M/1 queue with a non-homogeneous Poisson arrival process. The underlying dynamics of the arrival process cannot be formulated by a precise model. They derive dynamic pricing policies via a stochastic fluid model approximation. The solution is near-optimal for a market with high volumes of demand and slow variation of market size. Afèche and Ata [1] study an observable queue with two types of customers: patient ones with a lower waiting cost rate c_{low} and impatient ones with c_{high}. The proportion q of patient customers is unknown. They develop a Bayesian learning model using dynamic pricing in order to maximize the discounted expected revenue. Haviv and Randhawa [11] consider an $M/M/1$ unobservable queue with heterogeneous service values. They assume the system manager does not know about the arrival rate and applies demand-independent static pricing by solving a robust optimization problem. They find that the pricing policy performs well for some distributions of service value. Chen and Hasenbein [6] consider arrival rate uncertainty in Naor's model in both the observable and observable cases. Finally, some general results in Hassin and Snitkovsky [10] imply that Naor's results on thresholds hold for some models with arrival rate uncertainty.

1.3 Markov Decision Process Preliminaries

We follow the notation used by Puterman [18] and describe an MDP by a five-tuple $< T, S, A, P, R >$. T is a set of decision epochs or stages t at which the agent observes the system state and make decisions. S is the state space and S_t refers to the state at time t. A is the action set and $A_{s,t}$ denotes the set of allowable actions that can be taken after observing state s at time t. P represents the transition function, and $P_t(s' \mid s, a)$ defines the transition probability from s to s' by taking action a at time t. R is the reward function, and $r_t(s, a, s')$ defines the reward of transition from state s to s', by taking action a at time t. A deterministic and Markovian decision rule d maps S to A and specifies which action to take in state s. A deterministic and Markovian policy $\pi = \{d_1, d_2, \ldots\}$ is defined by a sequence of decision rules. When the same decision rule is applied at each decision epoch, we say that the policy is a stationary policy.

One can evaluate a policy with regard to the discounted reward criterion. Let $\gamma \in [0, 1)$ be a discount factor. The value of a policy, $V^\pi(s)$, is defined as the expected total discounted rewards starting at state s and following policy π. The discounted reward for an infinite-horizon MDP is

$$V = \sum_{i=1}^{\infty} \gamma^{i-1} r_i.$$

Where r_i is the reward obtained at step i. The optimal policy, π^*, is the policy that maximizes $V^\pi(s)$, $\forall s \in S$. The value V^π of policy π obeys the following equation:

$$V^\pi(s) = R(s, \pi(s)) + \gamma \sum_{s' \in S} P(s' \mid s, \pi(s)) V^\pi(s').$$

The Bellman equations characterize the optimal value function for a given MDP and criterion. It has the following formulation:

$$V^*(s) = \max_{a \in A} \left[R(s, a) + \gamma \sum_{s' \in S} P(s' \mid s, a) V^*(s') \right].$$

Existing algorithms for solving an MDP, including value iteration, policy iteration and linear programming, generally require accurate knowledge of transition probabilities. The optimal policy is sensitive to the transition probabilities. In practice, the transition probabilities are not known with certainty due to estimation errors. The uncertainty is described in terms of a set of possible transition probabilities. In some work, the problem is formulated in a Bayesian framework in which prior information on the transition probabilities is available. Some other work considers a game-theoretic formation using a max-min criterion under various uncertainty descriptions. The transition probabilities can be described by a set of linear inequalities (MDPIP) [22] or a convex set [2], or bounded by intervals (BMDP) [8, 19], or constrained by a "rectangularity property" [12, 16].

For an MDP with finite state and action space, the uncertain transition matrices can be described more specifically as follows. For each action, there exist a uncertainty set which includes all possible transition matrices associated with the action, and each row of a transition matrix describes the transitions from a single state to other states associated with the action. We consider a robust, i.e., adversarial model of the uncertain transition matrices. There are two primary ways to formulate the adversarial model [2, 12, 16]: a static adversary and a dynamic adversary. In the static model, nature chooses the same transition probability every time the same state-action pair is encountered, while in the dynamic model, nature chooses a possibly different transition probability at every time point. The static model may be reasonable as a conservative model of unknown parameters, but solving the problem is generally hard [2]. The dynamic model is a relaxation of the static model and can be solved efficiently by some algorithms.

Proposition 2 in [19] shows that there exists an optimal Markovian, stationary deterministic policy for the discounted cost dynamic adversarial model. Lemma 3.3 in [12] states that when the decision maker is restricted to stationary policies, then the objective values of static and dynamic adversarial models are the same.

1.4 Organization

In this research, we extend Naor's observable model in three ways: (1) by allowing customers to be economically heterogeneous, (2) by assuming that the arrival rate is not known with certainty, and (3), by considering state dependent pricing policies. We study three models which consider: (1)(2), (2)(3), and (1)(2)(3). The remainder of the dissertation is organized as follows. Chapter 2 considers a model with (1) and (2). Chapter 3 considers a model with (2) and (3). Chapter 4 considers a model with (1), (2) and (3). Chapter 5 summarizes the conclusions, our contributions and future work.

Chapter 2

Static Pricing with Uncertain Arrival Rate and Heterogeneous Customers

2.1 Model

The model we study is an $M/M/1$ system with an unknown arrival rate Λ and deterministic service rate μ, operating under the first-come-first-serve service discipline. Λ is a non-negative random variable with cdf F_Λ, which is known to customers and the system manager. We assume that an entering customer's waiting cost is a linear function of the total time he spends in the system. Customers have a non-negative service value X and waiting cost rate Y with proper joint cdf F_{XY}. Let F_X and F_Y be the marginal cdf's of X and Y, respectively. Further, we set $\mathbb{E}[X] = R$, $\mathbb{E}[Y] = C$, for consistency with the homogeneous case.

As mentioned above, the system manager does not observe individual customer values for service or waiting. A fixed admission fee p is imposed by the system manager. The queue length is observable to customers when they arrive. Customers are risk neutral and decide whether to enter the system or not upon arrival, based on their expected net benefit. Reneging is forbidden and a customers who balks receives zero net benefit.

Suppose a new customer arrives when the number of customers in the system is $n-1$ (for $n \geq 1$). His expected total waiting time is then $\frac{n}{\mu}$, including his own service. Let p be the admission fee, x be his service value and y be his waiting cost rate. Then the new customer enters the system if and only if his net benefit is non-negative, or $x - y\frac{n}{\mu} \geq p$. Now we wish to consider the fraction of customers that enter the system. First, define the random variables

$$\Theta_n = X - Y\frac{n}{\mu}, \quad n = 1, 2, \ldots.$$

Let F_{Θ_n} be the cdf of Θ_n, and let $\bar{F}_{\Theta_n}$ be the complementary cdf. Then the fraction of entering customers when the queue length is $n-1$, is given by

$$\bar{F}_{\Theta_n}(p) = P_{X,Y}\left(X - Y\frac{n}{\mu} \geq p\right), \quad n = 1, 2, \ldots. \tag{2.1}$$

By the splitting property of Poisson processes, the corresponding effective arrival rate in state $n-1$ is given by

$$\Lambda_{n-1}(p) = \Lambda \bar{F}_{\Theta_n}(p), \quad n = 1, 2, \ldots. \tag{2.2}$$

In the results in the sequel, we implicitly assume that both Λ and (X, Y) possess density functions. However, it should be clear from the proofs that the results also apply when these random variables are discrete.

2.2 Stability

In this section we study the stability of the model just introduced. Since the arrival rate is random, and the customer valuations are heterogeneous, establishing stability of the system is not completely trivial. In fact, considering the random arrival rate, established at time 0, we are really considering a *family* of $M/M/1$ queues. This motivates the next definition.

Definition 1. *Let $M^\Lambda/M/1$ denote an $M/M/1$ queue with a non-negative random arrival rate Λ. The system is said to be stable w.p. 1 if the associated Markov chain is positive recurrent w.p. 1.*

Now consider the $M^\Lambda/M/1$ system introduced in Section 2.1, with heterogeneous customers. For all $p \geq 0$, with $\bar{F}_{\Theta_n}(p)$ given in (2.1), define the following random variables:

$$\Phi_n(p) := \begin{cases} \frac{\Lambda^n}{\mu^n} \prod_{i=1}^n \bar{F}_{\Theta_i}(p) & n = 1, 2, \ldots \\ 1 & n = 0. \end{cases} \tag{2.3}$$

Theorem 2.2.1. *For a given value of $p \geq 0$, the $M^\Lambda/M/1$ with heterogeneous customers is stable w.p. 1 if and only if the following conditions hold w.p. 1:*

$$\begin{aligned} &\sum_{n=0}^{\infty} \Phi_n(p) < \infty \\ &\sum_{n=0}^{\infty} \frac{1}{\Lambda_n(p)\Phi_n(p)} = \infty. \end{aligned} \tag{2.4}$$

Proof. For a birth-death process with given birth rates $\{\lambda_0, \lambda_1, \ldots\}$ and death rates $\{\mu_1, \mu_2, \ldots\}$, set

$$\phi_n = \begin{cases} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} & n = 1, 2, \ldots \\ 1 & n = 0. \end{cases}$$

From standard results, the corresponding continuous-time Markov chain is positive recurrent iff

$$\sum_{n=0}^{\infty} \phi_n < \infty$$
$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n \phi_n} = \infty.$$

In our model, the effective arrival rate when the queue length is i is $\Lambda_i(p)$. Thus, for any $p \geq 0$ using (2.2) and (2.3) we have

$$\Phi_n(p) = \begin{cases} \frac{\prod_{i=0}^{n-1} \Lambda_i(p)}{\mu^n}, & n = 1, 2, \ldots \\ 1, & n = 0. \end{cases}$$

The result then holds by simply extending the classical results to our case. $\square$

Note that we employ a capital Φ_n for our case, since the multipliers are functions of Λ and hence are themselves random variables.

We now present an important sufficient condition for stability of our model.

Lemma 2.2.2. *For a fixed $p \geq 0$, the $M^\Lambda/M/1$ system with heterogeneous customers is stable w.p. 1 if*

$$\lim_{n \to \infty} \frac{\Lambda}{\mu} \bar{F}_{\Theta_n}(p) < 1 \quad \textit{w.p. 1.}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty}\Phi_n(p) &= 1+\sum_{n=1}^{\infty}\frac{\Lambda^n}{\mu^n}\prod_{i=1}^{n}\bar{F}_{\Theta_i}(p)\\ &= 1+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\left(\frac{\Lambda}{\mu}\bar{F}_{\Theta_i}(p)\right)\\ \sum_{n=0}^{\infty}\frac{1}{\Lambda_n(p)\Phi_n(p)} &= \frac{1}{\Lambda\bar{F}_{\Theta_1}(p)}+\sum_{n=1}^{\infty}\frac{\mu^n}{\Lambda^{n+1}\prod_{i=1}^{n+1}\bar{F}_{\Theta_i}(p)}\\ &= \frac{1}{\Lambda\bar{F}_{\Theta_1}(p)}+\sum_{n=1}^{\infty}\frac{1}{\mu\prod_{i=1}^{n+1}\left(\frac{\Lambda}{\mu}\bar{F}_{\Theta_i}(p)\right)}.\end{aligned}$$

For the first series, note that

$$\lim_{n\to\infty}\left|\frac{\prod_{i=1}^{n+1}\left(\frac{\Lambda}{\mu}\bar{F}_{\Theta_i}(p)\right)}{\prod_{i=1}^{n}\left(\frac{\Lambda}{\mu}\bar{F}_{\Theta_i}(p)\right)}\right| = \lim_{n\to\infty}\frac{\Lambda}{\mu}\bar{F}_{\Theta_{n+1}}(p)<1 \quad \text{w.p. 1.}$$

Then using the ratio test and the assumption of the theorem, we have

$$\sum_{n=0}^{\infty}\Phi_n(p)<\infty \quad w.p.\ 1.$$

For the second series, the ratio test is

$$\lim_{n\to\infty}\left|\frac{\mu\prod_{i=1}^{n}\left(\frac{\Lambda}{\mu}\bar{F}_{\Theta_i}(p)\right)}{\mu\prod_{i=1}^{n+1}\left(\frac{\Lambda}{\mu}\bar{F}_{\Theta_i}(p)\right)}\right| = \lim_{n\to\infty}\left[\frac{\Lambda}{\mu}\bar{F}_{\Theta_{n+1}}(p)\right]^{-1}>1 \quad \text{w.p. 1.}$$

We have used the fact that the expression in brackets has a limit strictly less 1. Hence, the limit of the inverse is strictly greater than 1. So, w.p. 1

$$\sum_{n=0}^{\infty}\frac{1}{\Lambda_n(p)\Phi_n(p)}=\infty.$$

Hence, the system is stable almost surely by Theorem 2.2.1. □

Lemma 2.2.3. *For an $M^{\Lambda}/M/1$ system with heterogeneous customers, $\forall p \geq 0$,*

$$\lim_{n\to\infty} \bar{F}_{\Theta_n}(p) = 0.$$

Proof. Let $f_Y(y)$ be the marginal pdf of Y and $\bar{F}_{X|y}$ be complementary conditional cdf of X (given $Y = y$). Then, using the law of total probability,

$$\begin{aligned}
\bar{F}_{\Theta_n}(p) &= P\left(X \geq p + \frac{n}{\mu}Y\right) \\
&= \int_0^{\infty} P\left(X \geq p + \frac{n}{\mu}Y \mid Y = y\right) f_Y(y)\mathrm{d}y \\
&= \int_0^{\infty} \bar{F}_{X|y}\left(\frac{n}{\mu}y + p\right) f_Y(y)\mathrm{d}y.
\end{aligned}$$

Clearly, $0 \leq \bar{F}_{X|y}\left(\frac{n}{\mu}y + p\right) \leq 1$ for all $n \geq 0$ and all $y \geq 0$. Thus, $\bar{F}_{X|y}$ is uniformly bounded by 1, which is integrable against $f_Y(y)$.

Applying dominated convergence, we obtain

$$\lim_{n\to\infty} \int_0^{\infty} \bar{F}_{X|y}\left(\frac{n}{\mu}y + p\right) f_Y(y)\mathrm{d}y = \int_0^{\infty} \lim_{n\to\infty} \bar{F}_{X|y}\left(\frac{n}{\mu}y + p\right) f_Y(y)\mathrm{d}y = 0,$$

for all $p \geq 0$. We used the fact that the limit inside the integral goes to zero, since $F_{X,Y}$ is assumed to be a proper cdf.

Thus $\forall p \geq 0$,

$$\lim_{n\to\infty} \bar{F}_{\Theta_n}(p) = 0.$$

$\square$

We are now prepared to prove our main stability result.

Theorem 2.2.4. *The $M^\Lambda/M/1$ with heterogeneous customers system is stable w.p. 1, for all $p \geq 0$.*

Proof. By Lemma 2.2.3, for all $p \geq 0$ and for each realization $\Lambda(\omega)$ of the arrival rate we have that

$$\lim_{n\to\infty} \frac{\Lambda(\omega)}{\mu} \bar{F}_{\Theta_n}(p) = \frac{\Lambda(\omega)}{\mu} \lim_{n\to\infty} \bar{F}_{\Theta_n}(p) = 0.$$

In other words, $\forall p \geq 0$,

$$\lim_{n\to\infty} \frac{\Lambda}{\mu} \bar{F}_{\Theta_n}(p) = 0 \quad \text{w.p. 1.}$$

Thus, by Lemma 2.2.2 the system is stable w.p. 1. □

2.3 Revenue Rate and Social Benefit Rate

In this section we study the behavior of the RM and SO, through their respective rate functions. In particular, for a fixed price p we analyze the revenue rate $Z(p)$ and the social benefit rate $S(p)$. These functions are the key to examining the relationship between the optimal admission fees for the the two types of optimizers.

With Φ_n defined in (2.3), for all $p \geq 0$, define the random variable

$$M = \sum_{n=0}^{\infty} \Phi_n = 1 + \sum_{n=1}^{\infty} \frac{\Lambda^n}{\mu^n} \prod_{i=1}^{n} \bar{F}_{\Theta_i}(p). \tag{2.5}$$

Recall that M is random via its dependence on Λ.

Next, for all $p \geq 0$, let $\pi_n(p)$ be the "expected" steady state probability that the number of customers in the system is n. Then

$$\pi_n(p) = E_\Lambda\left(\frac{\Phi_n}{M}\right) = \int_0^\infty \frac{1}{M(\lambda)}\Phi_n(\lambda) f_\Lambda(\lambda)\mathrm{d}\lambda, \quad n = 0, 1, 2, \ldots . \tag{2.6}$$

In the theorem below, note that we use the term "expected rate." For each realization of Λ, there is a revenue rate for any fixed p. When we average over all realizations of Λ, we then obtain an expected rate.

Theorem 2.3.1. *For an $M^\Lambda/M/1$ with heterogeneous customers, the expected revenue rate is given by*

$$Z(p) = \mu p\left(1 - \pi_0(p)\right).$$

The expected social benefit rate is given by

$$S(p) = \mu \sum_{i=1}^{\infty} \pi_i(p)\mathbb{E}\left[\Theta_i \mid \Theta_i \geq p\right].$$

Proof. By Theorem 2.2.4 the system is stable w.p. 1. For each realization of λ and each fixed p, the rate at which customers depart the system is $\mu(1 - \pi_0(p,\lambda))$, by standard results. Hence, the revenue rate is $\mu p(1 - \pi_0(p,\lambda))$. Taking expected values with respect to Λ yields the first result.

For the expected social benefit rate, first suppose there are i customers in the system, and a new customer arrives with service value x and waiting cost rate y. This customer will have an expected benefit of

$$\begin{cases} x - y\frac{i+1}{\mu} & \text{if} \quad x - y\frac{i+1}{\mu} \geq p \\ 0 & \text{if} \quad x - y\frac{i+1}{\mu} < p. \end{cases}$$

Of course, in the latter case the customer does not join the queue. Thus, in state i the expected benefit per customer is $E\left[\Theta_i \,|\, \Theta_i \geq p\right]$. Notice that this expectation is taken with respect to the cdf of Θ_i and is independent of Λ. Then, as argued above, we take the expectation with respect to Λ and compute the overall benefit rate by standard Markov chain theory to obtain $S = \mu \sum_{i=1}^{\infty} \pi_i(p) E\left[\Theta_i \,|\, \Theta_i \geq p\right]$. □

Finally, we end with a result pertaining to a general property of Z that will be of use in the next section. The proof of the theorem is given in the appendix. Note that the property is not immediately obvious, and requires additional conditions on the joint distribution of X and Y, and on Λ.

Theorem 2.3.2. *Suppose* $\mathbb{E}[\Theta_n^2] < \infty$ *for all* $n \geq 1$ *and that*

$$E\left[\frac{\Lambda/\mu}{1-\Lambda/\mu}\right] < \infty.$$

Then $\lim_{p\to\infty} Z(p) = 0$*.*

Note that $\mathbb{E}[\Theta_n^2]$ is finite for all n if and only if it is true for $n = 1$. For example, if X and Y each have a finite variance and jointly have a finite covariance, then the condition holds for all n. The second condition essentially says that the expected queue length in the corresponding, standard, $M/M/1$ queue must be finite. This second condition, although sufficient, may not be necessary. However, it is needed in our proof to carry out several applications of dominated convergence.

2.4 Heterogeneous Service Values

Unfortunately, the expressions for the expected revenue and social benefit rates given in Theorem 2.3.1 are somewhat complicated to analyze in the general case. Below, we give an example in which the social benefit rate can be simplified allowing a more nuanced comparison of the SO and RM.

In particular, consider the case where the service value is exponentially distributed and the waiting cost rate is constant. Suppose then that $X \sim \exp(\frac{1}{R})$ and $Y \equiv C$.

Theorem 2.4.1. *Consider an $M^\Lambda/M/1$ system with heterogeneous customers. If the service value $X \sim \exp(\frac{1}{R})$ and the waiting cost rate is C, then the social benefit rate is given by*

$$S(p) = \mu(R+p)\left(1 - \pi_0(p)\right).$$

Proof. Using the memoryless property we have,

$$E\left[\Theta_i \,|\, \Theta_i \geq p\right] = E\left[X - \frac{Ci}{\mu} \,\middle|\, X \geq p + \frac{Ci}{\mu}\right] = R + p.$$

Thus

$$\begin{aligned} S(p) &= \mu \sum_{i=1}^{\infty} \pi_i(p) E\left[\Theta_i \,|\, \Theta_i \geq p\right] \\ &= \mu(R+p) \sum_{i=1}^{\infty} \pi_i(p) \\ &= \mu(R+p)\left(1 - \pi_0(p)\right). \end{aligned}$$

□

With this simplification in hand, we can say more about the relationship between the optimal fees for the SO and RM.

Theorem 2.4.2. *Consider an $M^{\Lambda}/M/1$ system with heterogeneous customers, service value $X \sim \exp(\frac{1}{R})$, and $Y \equiv C$. Assume*

$$E\left[\frac{\Lambda/\mu}{1-\Lambda/\mu}\right] < \infty.$$

Suppose the social benefit rate $S(p)$ and the revenue rate $Z(p)$ are unimodal and continuously differentiable on $p \in [0, \infty)$. Let p_Z^* and p_S^* be the social-optimizing and revenue-maximizing fees, respectively. Then

$$p_S^* \leq p_Z^*.$$

Proof. First, we have

$$S(p) = \mu(R+p)\left(1-\pi_0(p)\right)$$

and

$$S'(p) = \mu\left(1-\pi_0(p)\right) - \mu(R+p)\pi_0'(p).$$

Similarly,

$$Z(p) = \mu p\left(1-\pi_0(p)\right) \tag{2.7}$$

and

$$Z'(p) = \mu\left(1-\pi_0(p)\right) - \mu p \pi_0'(p). \tag{2.8}$$

Since p_S^* maximizes $S(p)$,

$$S'(p_S^*) = \mu\left(1-\pi_0(p_S^*)\right) - \mu(R+p_S^*)\pi_0'(p_S^*) = 0. \tag{2.9}$$

Set $g(p) = \frac{\mathrm{d}Z(p)}{\mathrm{d}p}$. Then combining (2.7), (2.8) and (2.9) we have

$$g(p_S^*) = \mu(R + p_S^*)\pi_0'(p_S^*) - \mu p_S^* \pi_0'(p_S^*) = \mu R \pi_0'(p_S^*).$$

We now use a modification of the arguments used to prove Lemma 1.0.1 in the appendix. Note that for each i, the $\bar{F}_{\Theta_i}$ in this special case is strictly decreasing, since it is the ccdf of a shifted exponential random variable. Hence, $\pi_0(p)$ is a strictly increasing function of p on $[0, \infty)$. So,

$$\pi_0'(p) > 0 \quad \forall p > 0.$$

Thus, in particular $g(p_S^*) > 0$.

Now, suppose $p_S^* > p_Z^*$, by assumption p_Z^* is the unique maximum of Z. However, we have that $g(p_S^*) > 0$ and that there must exist another point $\tilde{p} > p_S^*$, with $g(\tilde{p}) < 0$. We know this is the case because $\lim_{p\to\infty} Z(p) = 0$ by Theorem 2.3.2, which implies that the derivative of Z must be negative for arbitrarily large values of p. These two facts about the derivative of Z imply the existence of another local minimum, which contradicts our unimodality assumption. Hence $p_S^* \leq p_Z^*$. $\square$

The unimodality assumption is critical here. Numerical evidence indicates that S and Z are indeed unimodal in this case, but establishing these properties directly is quite difficult, due to the complex expression for π_0.

2.5 Computational Study

In this section, we give some examples of interest, in which we see the revenue rate and the social benefit rate are not unimodal functions of the admission fee.

2.5.1 Two-category Service

In this section, we consider a model in which the service value is a discrete random variable with two possible values, but the waiting cost rate is a constant. Suppose

$$X = \begin{cases} R_l & \text{w.p.} \quad q_l \\ R_h & \text{w.p.} \quad q_h. \end{cases}$$

Assume $0 \le R_l \le R_h$, and $q_l + q_h = 1$. The waiting cost rate $Y \equiv C$.

First, we compute the joining fraction of the customers. Define

$$K_1 = \left\lfloor \frac{\mu(R_l - p)}{C + \epsilon} \right\rfloor$$

and

$$K_2 = \left\lfloor \frac{\mu(R_h - p)}{C - \epsilon} \right\rfloor.$$

When $K_1 \ge 1$,

$$R_l - C\frac{i}{\mu} \ge p, \quad i = 1, 2, \ldots, K_1.$$

The joining fraction of customers, when the queue length is $i-1$, is

$$\begin{aligned}\bar{F}_{\Theta_i}(p) &= \Pr\{X - C\frac{i}{\mu} \geq p\} \\ &= \Pr\{X \geq C\frac{i}{\mu} + p\} \\ &= \begin{cases} 1 & 1 \leq i \leq K_1 \\ q_h & K_1+1 \leq i \leq K_2 \\ 0 & i \geq K_2+1. \end{cases}\end{aligned}$$

When $K_1 < 1$, $K_2 \geq 1$,

$$R_h - C\frac{i}{\mu} \geq p, \quad i = 1, 2, \ldots, K_2.$$

Then

$$\bar{F}_{\Theta_i}(p) = \begin{cases} q_h & 1 \leq i \leq K_2 \\ 0 & i \geq K_2+1. \end{cases}$$

When $K_2 < 1$,

$$R_h - C\frac{i}{\mu} < p, \quad i \geq 1.$$

In this case, no customer is willing to enter the system.

Next, we compute the steady-state probabilities. $\forall p \geq 0$, if $K_1 \geq 1$,

$$\Phi_n = \begin{cases} 1 & n = 0 \\ \frac{\Lambda^n}{\mu^n} & 1 \leq n \leq K_1 \\ \frac{\Lambda^n}{\mu^n} q_h^{n-K_1} & K_1+1 \leq n \leq K_2 \\ 0 & n \geq K_2+1. \end{cases}$$

When $K_1 < 1$, $K_2 \geq 1$,

$$\Phi_n = \begin{cases} 1 & n = 0 \\ \frac{\Lambda^n}{\mu^n} q_h^n & 1 \leq n \leq K_2 \\ 0 & n \geq K_2+1. \end{cases}$$

When $K_2 < 1$,

$$\Phi_n = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1. \end{cases}$$

According to (2.5),

$$\begin{aligned} M &= 1 + \sum_{n=1}^{K_2} \Phi_n \\ &= \begin{cases} 1 + \sum_{n=1}^{K_1} \frac{\Lambda^n}{\mu^n} + \sum_{n=K_1+1}^{K_2} \frac{\Lambda^n}{\mu^n} p_h^{n-K_1} & K_1 \geq 1 \\ 1 + \sum_{n=1}^{K_2} \frac{\Lambda^n}{\mu^n} p_h^n & K_1 < 1, K_2 \geq 1 \\ 1 & K_2 < 1. \end{cases} \end{aligned}$$

Then using (2.6), the steady-state probabilities are

$$\begin{aligned} \pi_0(p) &= E_\Lambda \left(\frac{1}{M} \right) \\ \pi_n(p) &= E_\Lambda \left(\frac{\Phi_n}{M} \right). \end{aligned}$$

Next, in order to get the social benefit rate, we also need the expected benefit of a single customer given he decides to enter the system:

$$E\left[\Theta_i \,|\, \Theta_i \geq p\right] = \begin{cases} R_l q_l + R_h q_h - \frac{C}{\mu} i & 0 \leq p < R_l - \frac{C}{\mu} i \\ R_h - \frac{C}{\mu} i & R_l - \frac{C}{\mu} i \leq p < R_h - \frac{C}{\mu} i \\ 0 & p \geq R_h - \frac{C}{\mu} i. \end{cases}$$

Thus, we can compute the revenue rate and the social benefit rate according to Theorem 2.3.1.

Finally, we give four computational examples. In case 1 and 2, we assume a discrete arrival rate with same possible values but different probabilities. In case 3 and 4 we assume an exponentially distributed arrival rate with different expectations. The parameters are shown in Table 2.1.

Two-category service Constant cost	λ	μ	R_l	R_h	C
Case 1	[0.5,5] w.p. [0.1,0.9]	1	50	150	5
Case 2	[0.5,5] w.p. [0.9,0.1]	1	50	150	5
Case 3	$\exp(\frac{1}{0.95})$	1	50	150	5
Case 4	$\exp(\frac{1}{4.5})$	1	50	150	5

Table 2.1: Two-category Service Value, Constant Waiting Cost Rate

We investigate the revenue rate and the social benefit rate in these cases. We present the revenue rate in figures 2.1, 2.2, 2.3 and 2.4. These four plots indicate that the revenue rate, as a function of the admission fee p, can have more than one modes. Similarly, we show the social benefit rate in figures 2.5, 2.6, 2.7 and 2.8. The social benefit rate can also have multiple modes in figures 2.5 and 2.7.

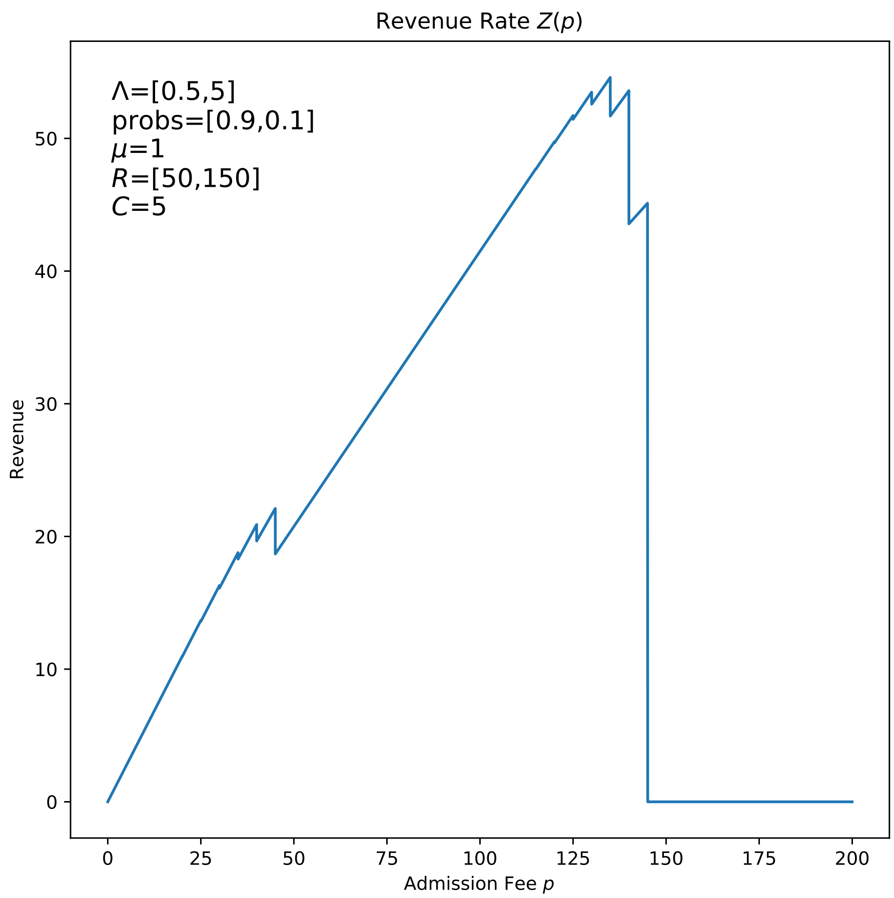

Figure 2.1: Two-category Service Value, Constant Waiting Cost Rate case 1, Revenue Rate

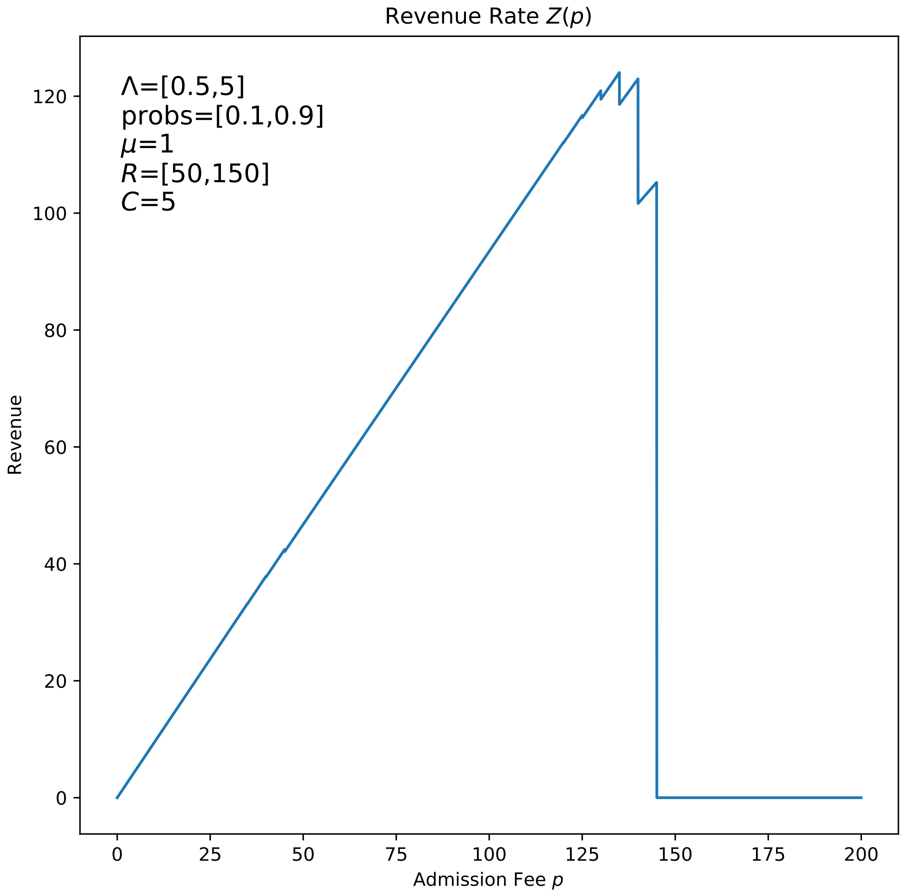

Figure 2.2: Two-category Service Value, Constant Waiting Cost Rate, case 2, Revenue Rate

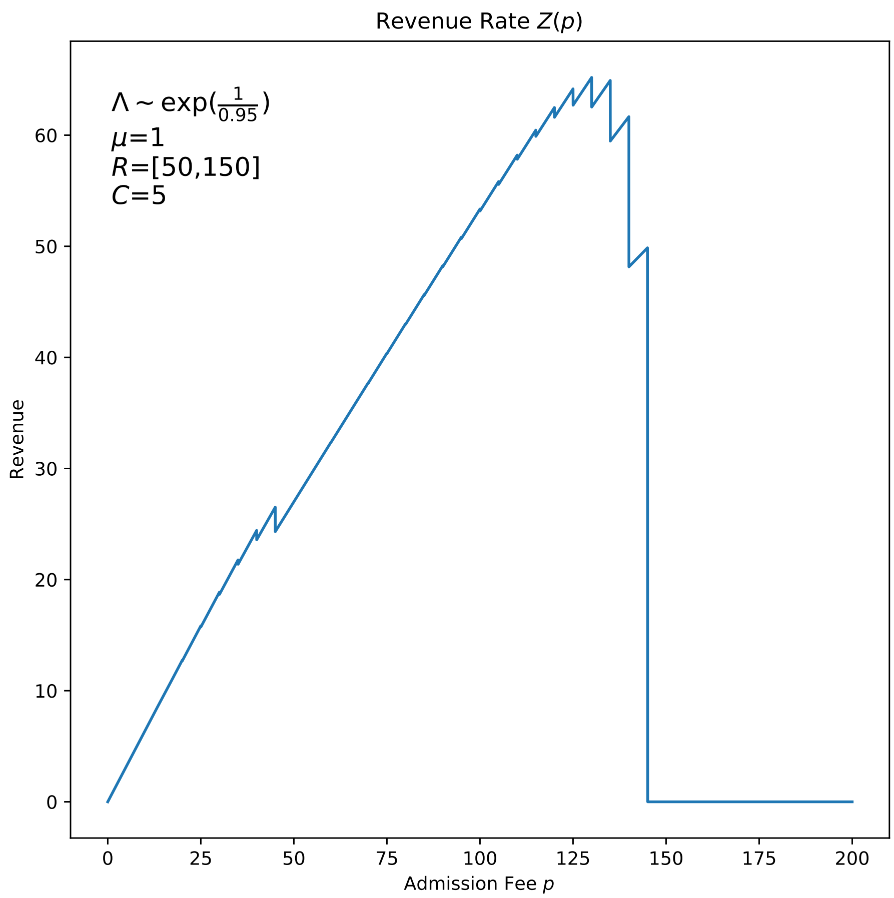

Figure 2.3: Two-category Service Value, Constant Waiting Cost Rate, case 3, Revenue Rate

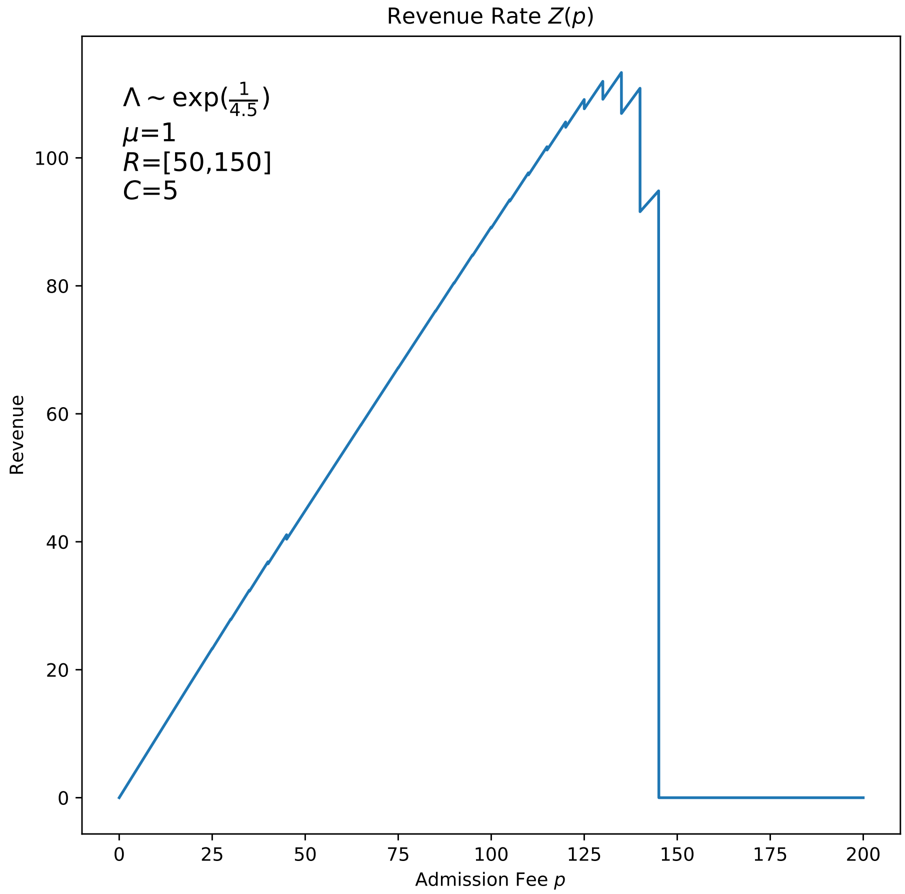

Figure 2.4: Two-category Service Value, Constant Waiting Cost Rate, case 4, Revenue Rate

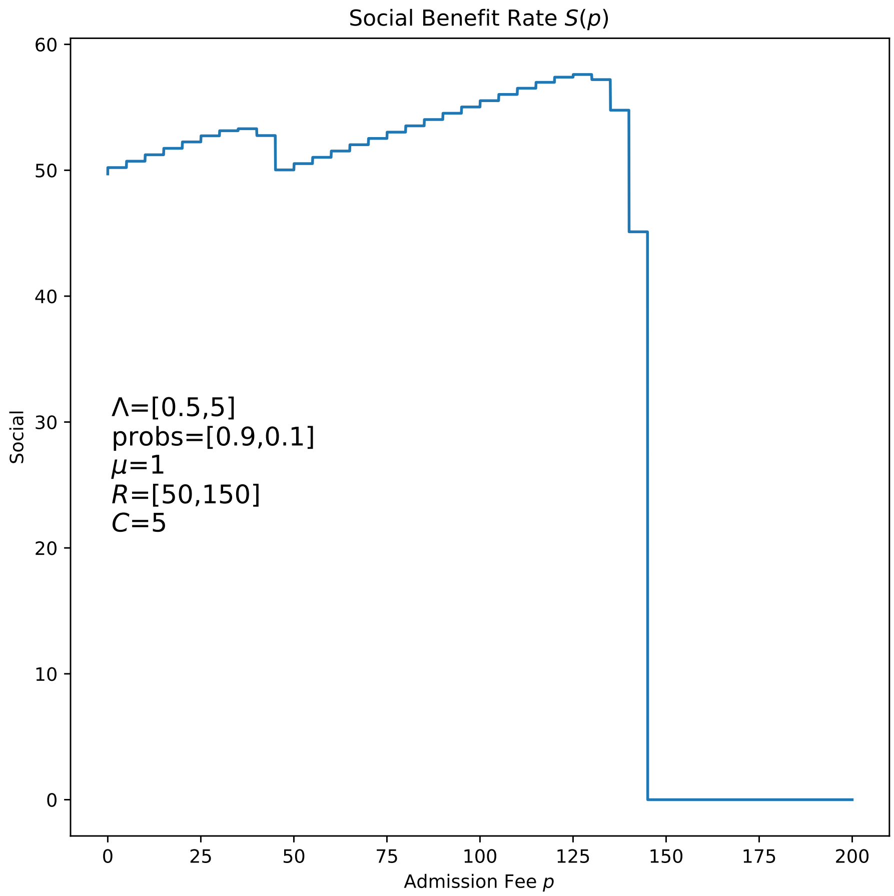

Figure 2.5: Two-category Service Value, Constant Waiting Cost Rate, case 1, Social Benefit Rate

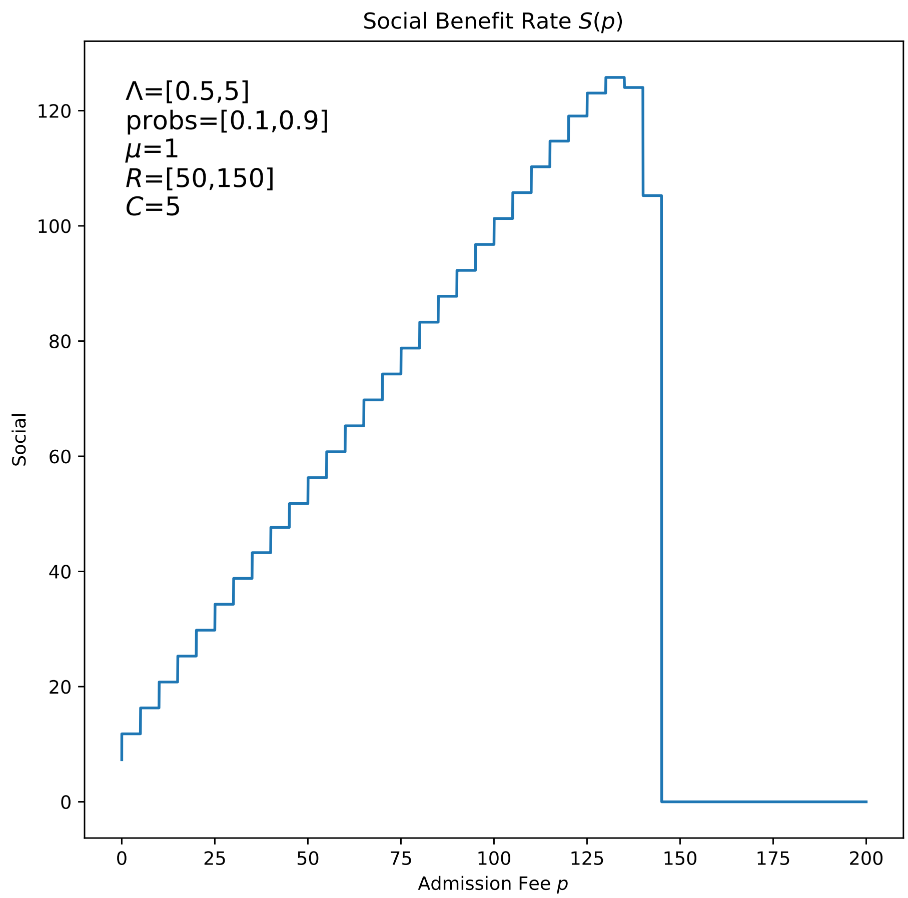

Figure 2.6: Two-category Service Value, Constant Waiting Cost Rate, case 2, Social Benefit Rate

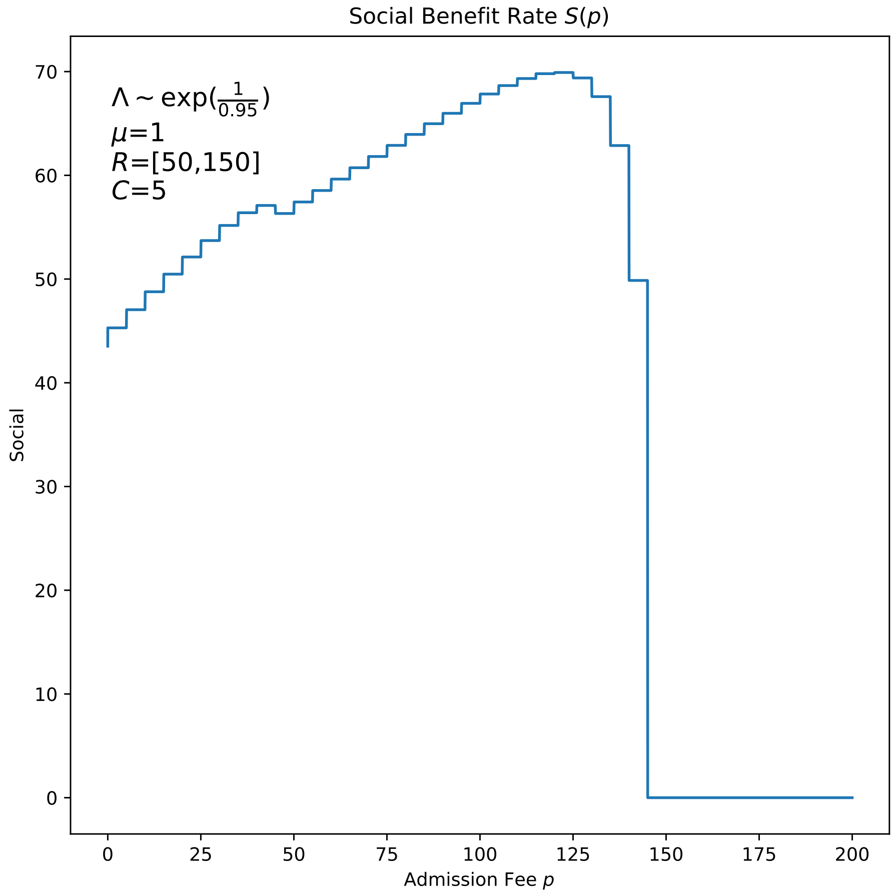

Figure 2.7: Two-category Service Value, Constant Waiting Cost Rate, case 3, Social Benefit Rate

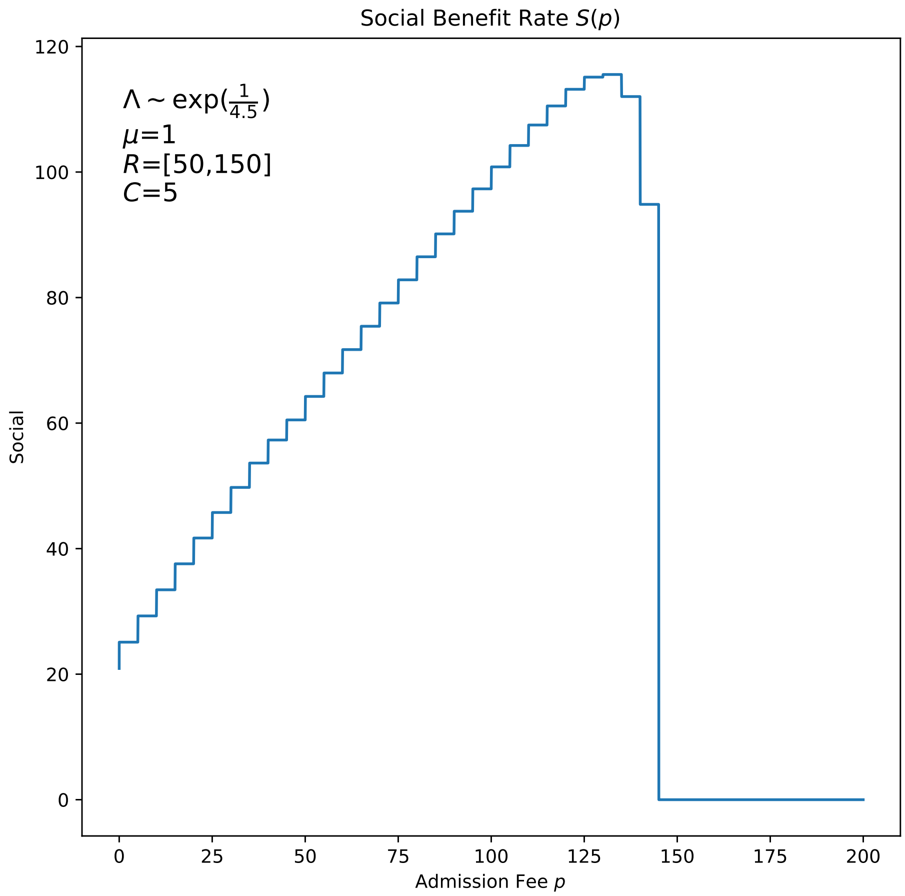

Figure 2.8: Two-category Service Value, Constant Waiting Cost Rate, case 4, Social Benefit Rate

2.5.2 Uniform Waiting Cost

In this section, we give a computational example in which customers have a fixed service value, but different waiting cost rates. Suppose the service value is constant with $X \equiv R$ and let the waiting cost be uniformly distributed, with $Y \sim U[C-\epsilon, C+\epsilon]$, where $\epsilon \in (0, C)$.

First, we compute the joining fraction of the customers. Define

$$K_1 = \left\lfloor \frac{\mu(R-p)}{C+\epsilon} \right\rfloor$$

and

$$K_2 = \left\lfloor \frac{\mu(R-p)}{C-\epsilon} \right\rfloor.$$

When $K_1 \geq 1$,

$$R - (C+\epsilon)\frac{i}{\mu} \geq p, \quad i = 1, 2, \ldots, K_1.$$

The joining fraction of customers, when the current queue length is $i-1$, is

$$\begin{aligned} \bar{F}_{\Theta_i}(p) =& \Pr\{R - Y\frac{i}{\mu} \geq p\} \\ =& P\left(Y \leq \frac{\mu(R-p)}{i}\right) \\ =& \begin{cases} 1 & 1 \leq i \leq K_1 \\ \frac{\mu(R-p)}{2\epsilon i} - \frac{C-\epsilon}{2\epsilon} & K_1 + 1 \leq i \leq K_2 \\ 0 & i \geq K_2 + 1. \end{cases} \end{aligned}$$

When $K_1 < 1$, $K_2 \geq 1$,

$$R - (C-\epsilon)\frac{i}{\mu} \geq p, \quad i = 1, 2, \ldots, K_2.$$

Then

$$\bar{F}_{\Theta_i}(p) = \begin{cases} \frac{\mu(R-p)}{2\epsilon i} - \frac{C-\epsilon}{2\epsilon} & 1 \le i \le K_2 \\ 0 & i \ge K_2 + 1. \end{cases}$$

When $K_2 < 1$,

$$R - (C-\epsilon)\frac{i}{\mu} < p, \quad i \ge 1.$$

Again, in this case, no customer is willing to enter the system.

Next, we compute the steady-state probabilities. $\forall p \ge 0$, if $K_1 \ge 1$,

$$\Phi_n = \begin{cases} 1 & n = 0 \\ \frac{\Lambda^n}{\mu^n} & 1 \le n \le K_1 \\ \frac{\Lambda^n}{\mu^n} \prod_{i=K_1+1}^{n} \left[\frac{\mu(R-p)}{2\epsilon i} - \frac{C-\epsilon}{2\epsilon}\right] & K_1 + 1 \le n \le K_2 \\ 0 & n \ge K_2 + 1. \end{cases}$$

When $K_1 < 1$, $K_2 \ge 1$,

$$\Phi_n = \begin{cases} 1 & n = 0 \\ \frac{\Lambda^n}{\mu^n} \prod_{i=1}^{n} \left[\frac{\mu(R-p)}{2\epsilon i} - \frac{C-\epsilon}{2\epsilon}\right] & 1 \le n \le K_2 \\ 0 & n \ge K_2 + 1. \end{cases}$$

When $K_2 < 1$,

$$\Phi_n = \begin{cases} 1 & n = 0 \\ 0 & n \ge 1. \end{cases}$$

According to (2.5),

$$\begin{aligned} M &= 1 + \sum_{n=1}^{K_2} \Phi_n \\ &= \begin{cases} 1 + \sum_{n=1}^{K_1} \frac{\Lambda^n}{\mu^n} + \sum_{n=K_1+1}^{K_2} \frac{\Lambda^n}{\mu^n} \prod_{i=K_1+1}^{n} \left[\frac{\mu(R-p)}{2\epsilon i} - \frac{C-\epsilon}{2\epsilon}\right] & K_1 \ge 1 \\ 1 + \sum_{n=1}^{K_2} \frac{\Lambda^n}{\mu^n} \prod_{i=K_1+1}^{n} \left[\frac{\mu(R-p)}{2\epsilon i} - \frac{C-\epsilon}{2\epsilon}\right] & K_1 < 1, K_2 \ge 1 \\ 1 & K_2 < 1. \end{cases} \end{aligned}$$

Then using (2.6), the steady-state probabilities are

$$\pi_0(p) = E_\Lambda\left(\frac{1}{M}\right)$$
$$\pi_n(p) = E_\Lambda\left(\frac{\Phi_n}{M}\right).$$

Next, in order to get the social benefit rate, we also need the expected benefit of a single customer give he decides to enter the system. Some algebra yields:

$$E\left[\Theta_i \,|\, \Theta_i \geq p\right] = \begin{cases} R - \frac{C}{\mu} i & 0 \leq p < R - \frac{C+\epsilon}{\mu} i \\ \frac{1}{2}\left(R + p - \frac{C-\epsilon}{\mu} i\right) & R - \frac{C+\epsilon}{\mu} i \leq p < R - \frac{C-\epsilon}{\mu} i \\ 0 & p \geq R - \frac{C-\epsilon}{\mu} i. \end{cases}$$

Thus, we can compute the revenue rate and the social benefit rate according to Theorem 2.3.1.

Finally, we give two computational examples. We assume a discrete arrival rate in the first case, and an exponential arrival rate in the second case. The parameters are shown in Table 2.2.

Two-category service Constant cost	λ	μ	R	C	ϵ
case 1	[0.1,5,10.6] w.p. [0.3,0.4,0.3]	1	100	10	1
case 2	$\exp(\frac{1}{0.5})$	1	100	10	1

Table 2.2: Constant Service Value, Uniform Waiting Cost Rate

The revenue rate functions are presented in figures 2.9 and 2.10. In the discrete arrival rate case, indicated by figure 2.9, the revenue rate function has multiple modes in terms of p. In the exponential arrival rate case, the arrival rate values are sampled from the given distribution. The revenue rate function seems to have multiple modes (figure 2.10). The social benefit rate functions are given in figures 2.11 and 2.12. The social benefit rate, when the arrival rate is discrete, has multiple modes (figure 2.11).

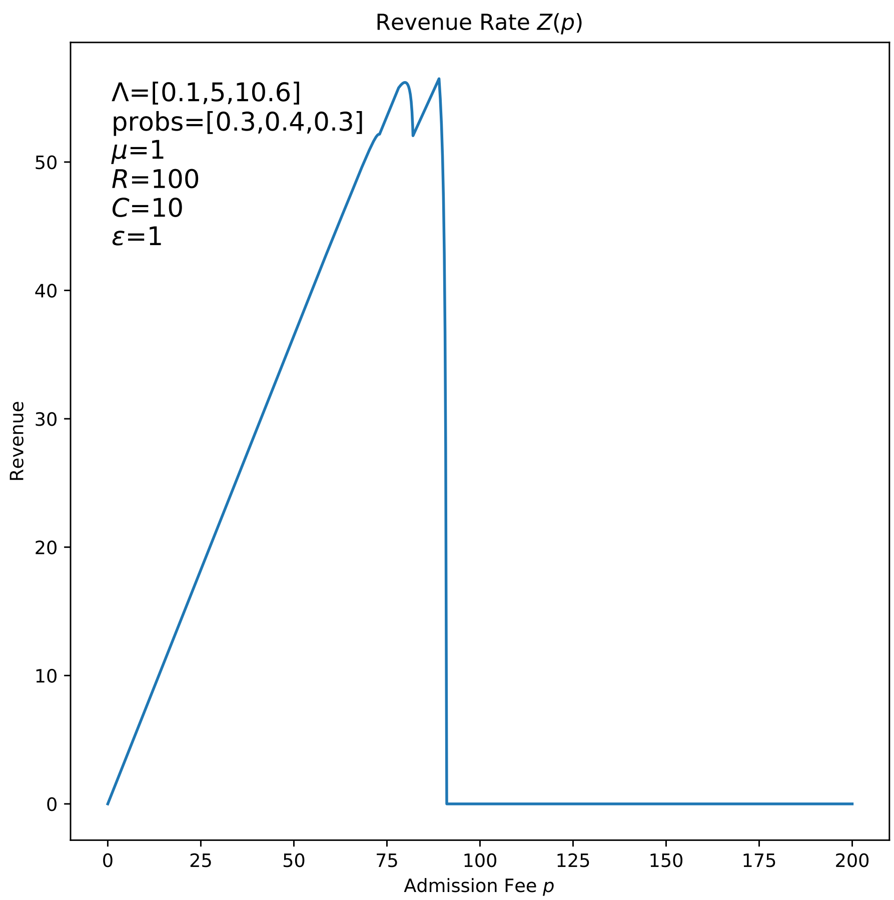

Figure 2.9: Constant Service Value, Uniform Waiting Cost Rate, case 1, Revenue Rate

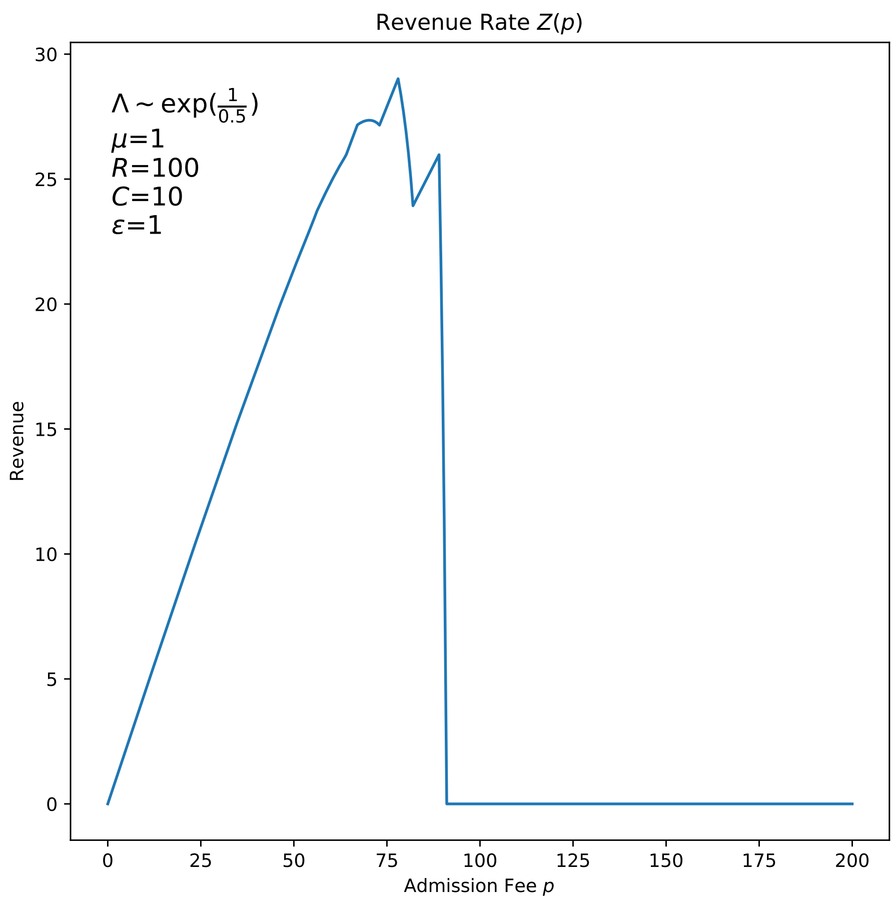

Figure 2.10: Constant Service Value, Uniform Waiting Cost Rate, case 2, Revenue Rate

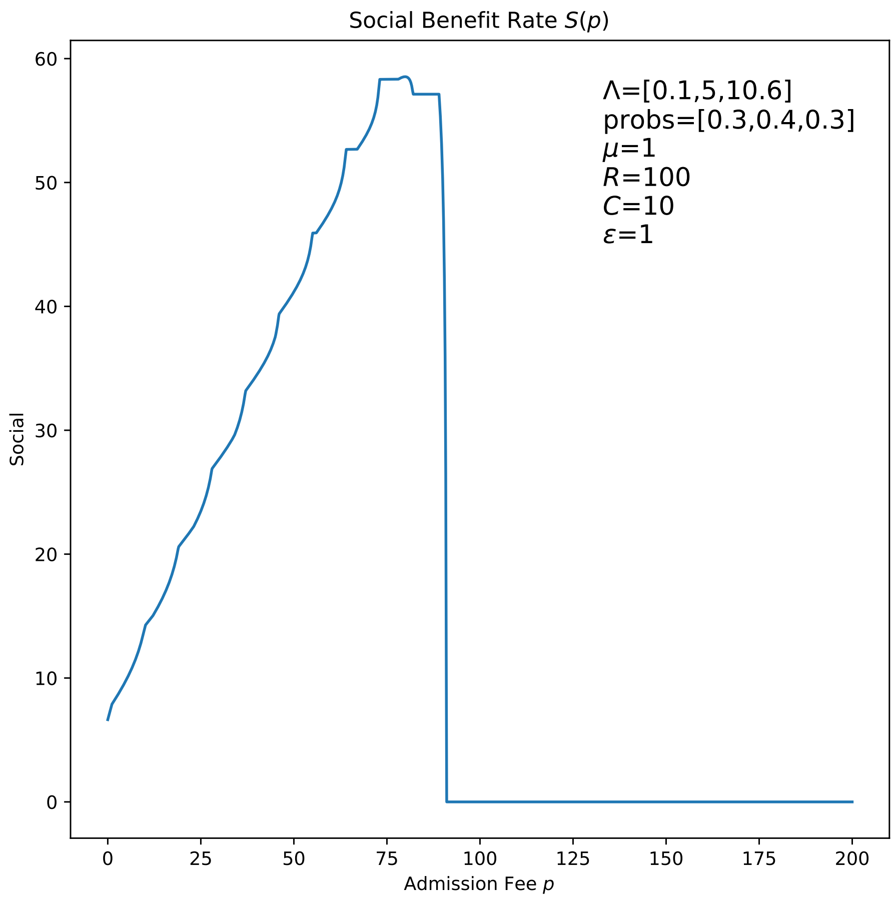

Figure 2.11: Constant Service Value, Uniform Waiting Cost Rate, case 1, Social Benefit Rate

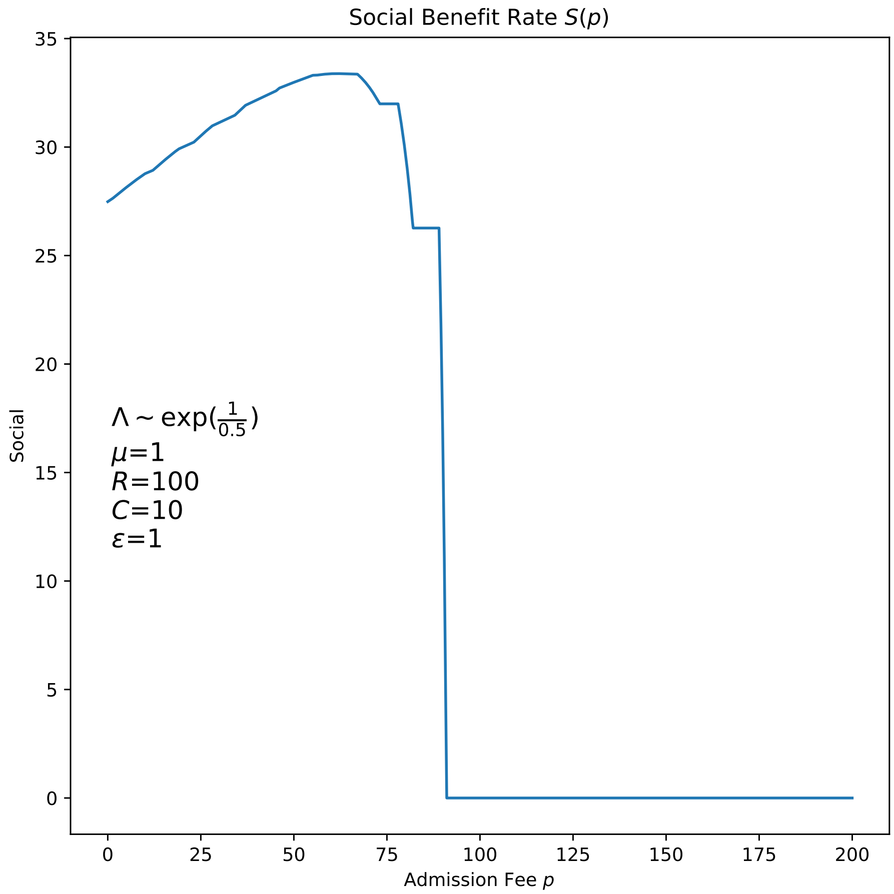

Figure 2.12: Constant Service Value, Uniform Waiting Cost Rate, case 2, Social Benefit Rate

Chapter 3

State-dependent Pricing with Uncertain Arrival Rate and Homogeneous Customers

In this chapter we study a model with similar settings in Chapter 2. Some different assumptions include: the unknown arrival rate Λ is chosen by nature from a given set $[\underline{\lambda}, \bar{\lambda}]$, where $0 < \underline{\lambda} \leq \bar{\lambda}$; Customers have identical values of service valuation R and waiting cost rate C; The admission fee may change according to the queue length; Customers and the system manager discount benefits and costs using a discount factor γ, with $0 < \gamma < 1$.

3.1 Individual Customer Decision

Suppose the queue length is n, , $n \geq 0$ and the admission fee is p_n. There is a new arrival and let τ be his waiting time in the system. Then τ has a gamma distribution with shape parameter $n+1$ and scale parameter μ, and $E_\tau(e^{-\gamma\tau}) = \phi^{n+1}$, where $\phi = \frac{\mu}{\mu+\gamma}$. The customer enters the system if and only if his expected net benefit is non-negative:

$$E_\tau\left[e^{-\gamma\tau}R - p_n - C\int_0^\tau e^{-\gamma t}dt\right] = \phi^{n+1}\left(R + \frac{C}{\gamma}\right) - \left(p_n + \frac{C}{\gamma}\right) \geq 0.$$

For all $n \geq 0$, the maximum admission fee inducing a new customer to enter the system is defined as $\bar{p}_n$, and

$$
\begin{aligned}
\bar{p}_n &= \left[\phi^{n+1}\left(R+\frac{C}{\gamma}\right)-\frac{C}{\gamma}\right]^+ \\
&= \begin{cases} \phi^{n+1}\left(R+\frac{C}{\gamma}\right)-\frac{C}{\gamma} & \phi^{n+1}\left(R+\frac{C}{\gamma}\right)-\frac{C}{\gamma}\geq 0 \\ 0 & \phi^{n+1}\left(R+\frac{C}{\gamma}\right)-\frac{C}{\gamma}\leq 0. \end{cases}
\end{aligned}
\tag{3.1}
$$

For all $p_n \geq 0$, define

$$
\hat{n}(p_n) = \left\lfloor \log\left(\frac{p_n + C/\gamma}{R + C/\gamma}\right) / \log\phi \right\rfloor . \tag{3.2}
$$

A new customer enters the system if the current queue length $n \leq \hat{n}(p_n) - 1$, and balks if $n \geq \hat{n}(p_n)$.

The maximum possible queue length is

$$
\bar{n} = \left\lfloor \log\left(\frac{C/\gamma}{R + C/\gamma}\right) / \log\phi \right\rfloor . \tag{3.3}
$$

3.2 Transition Probabilities

The original problem of the SO or the RM can be modeled as a DTMDP, in which the system state is the queue length. We utilize uniformization and model the system as an infinite-horizon DTMDP. The state space is discrete and finite, denoted by $S = \{0, 1, \cdots, \bar{n}\}$, where $\bar{n}$ is defined in (3.3). The transition rates are

$$
v_n = \begin{cases} \Lambda & n = 0 \\ \Lambda + \mu & 1 \leq n \leq \hat{n}(p_n) - 1 \\ \mu & n \geq \hat{n}(p_n). \end{cases}
$$

The transition probabilities are

$$
\begin{aligned}
q_{n,n-1} &= \frac{\mu}{\Lambda+\mu} \quad n \geq 1 \\
q_{n,n+1} &= \begin{cases} \frac{\Lambda}{\Lambda+\mu} & 1 \leq n \leq \hat{n}(p_n)-1 \\ 1 & n=0 \end{cases} \\
q_{nn} &= 0.
\end{aligned}
$$

Let $v = \bar{\lambda} + \mu$. We convert the CTMDP into an infinite horizon DTMDP.

The new discount factor is

$$
\alpha = \frac{v}{\gamma + v}.
$$

The new transition probabilities are

$$
\begin{aligned}
q_{n,n-1} &= \frac{\mu}{v} \quad n \geq 1 \\
q_{n,n+1} &= \begin{cases} \frac{\Lambda}{v} & 0 \leq n \leq \hat{n}(p_n)-1 \\ 0 & n \geq \hat{n}(p_n) \end{cases} \\
q_{nn} &= \begin{cases} \frac{v-\Lambda}{v} & n=0 \\ \frac{v-\Lambda-\mu}{v} & 1 \leq i \leq \hat{n}(p_n)-1 \\ \frac{v-\mu}{v} & n \geq \hat{n}(p_n). \end{cases}
\end{aligned}
$$

3.3 Robust Revenue Rate

In this section we study the RM's problem with the objective of maximizing the total discounted revenue rate. By the assumption on the arrival rate, nature chooses a value for Λ at time 0, and it remains fixed but not observable to either customers or the RM. The original problem is a static adversarial model. We first allow nature to independently draw a λ from the given set $[\underline{\lambda}, \bar{\lambda}]$ at every decision point. We consider the max-min criterion in order to find an optimal pricing policy.

First, we derive the Bellman equations for the RM's problem. Let $V^*(n)$ be the optimal revenue rate starting from queue length n, $n \geq 0$. Then $V^*(n)$ is the unique solution of the following robust Bellman equations:

$$
\begin{aligned}
V(0) &= \frac{1}{\gamma + v} \max_{p_0} \min_{\Lambda} \Big\{ vV(0) + \Lambda \left[p_0 + V(1) - V(0) \right] \Big\} \\
V(n) &= \frac{1}{\gamma + v} \max_{p_n} \min_{\Lambda} \Big\{ \mu 1_{\{n \geq 1\}} V(n-1) \\
&\quad + (v - \Lambda - \mu) 1_{\{1 \leq n < \hat{n}(p_n)\}} V(n) + (v - \mu) 1_{\{n \geq \hat{n}(p_n)\}} V(n) \\
&\quad + \Lambda 1_{\{1 \leq n < \hat{n}(p_n)\}} \left[p_n + V(n+1) \right] \Big\} \qquad n = 1, 2, \ldots.
\end{aligned}
$$

Then

$$
V(0) = \frac{v}{\gamma + v} V(0) + \frac{1}{\gamma + v} \max_{p_0} \min_{\Lambda} \Lambda \left[p + V(1) - V(0) \right].
$$

$$
\begin{aligned}
V(n) = &\frac{1}{\gamma+v}\max_{p_n}\min_{\Lambda}\Big\{\mu 1_{\{1\le n<\hat{n}(p_n)\}}V(n-1)+\mu 1_{\{n\ge\hat{n}(p_n)\}}V(n-1)\\
&\quad +(v-\Lambda-\mu)1_{\{1\le n<\hat{n}(p_n)\}}V(n)+(v-\mu)1_{\{n\ge\hat{n}(p_n)\}}V(n)\\
&\quad +\Lambda 1_{\{1\le n<\hat{n}(p_n)\}}\left[p_n+V(n+1)\right]\Big\}\\
= &\frac{1}{\gamma+v}\max_{p_n}\min_{\Lambda}\Big\{\left[\mu V(n-1)+(v-\mu)V(n)\right]1_{\{n\ge\hat{n}(p_n)\}}\\
&\quad \left[\mu V(n-1)+(v-\mu)V(n)+\Lambda\big(p_n+V(n+1)-V(n)\big)\right]1_{\{1\le n<\hat{n}(p_n)\}}\Big\}\\
= &\frac{1}{\gamma+v}\max_{p_n}\Big\{\left[\mu V(n-1)+(v-\mu)V(n)\right]1_{\{n\ge\hat{n}(p_n)\}}\\
&\left[\mu V(n-1)+(v-\mu)V(n)+\min_{\Lambda}\Lambda\big(p_n+V(n+1)-V(n)\big)\right]1_{\{1\le n<\hat{n}(p_n)\}}\Big\}\\
= &\frac{1}{\gamma+v}\Big\{\left[\mu V(n-1)+(v-\mu)V(n)\right]\\
&\quad +\max_{p_n}\min_{\Lambda}\Lambda\left(p_n+V(n+1)-V(n)\right)1_{\{1\le n<\hat{n}(p_n)\}}\Big\}\\
= &\frac{1}{\gamma+v}\Big\{\left[\mu V(n-1)+(v-\mu)V(n)\right]\\
&\quad +\max_{p_n}\left[\min_{\Lambda}\Lambda\left[p_n+V(n+1)-V(n)\right]\right]^{+}\Big\}\\
&\quad\ n=1,2,\ldots.
\end{aligned}
$$

If p_n is chosen so that an arriving customer is not willing to enter the system, then $n\ge\hat{n}(p_n)$ and $V(n)$ does not depend on p_n and Λ. If p_n is chosen so that an arriving customer is willing to enter the system, then $n<\hat{n}(p_n)$. Thus let p_n^* be the fee satisfying

$$\hat{n}(p_n^*)=n+1.$$

According to (3.1),

$$p_n^*=\phi^{n+1}\left(R+\frac{C}{\gamma}\right)-\frac{C}{\gamma}. \tag{3.4}$$

In order to maximize the revenue rate, the RM charges the largest p_n such that $n < \hat{n}(p_n)$, which yields $p_n = p_n^*$. The robust Bellman equations are then simplified as

$$\begin{aligned} V(0) &= \frac{1}{\gamma+v}\Big\{ vV(0) + \underline{\lambda}\,[p_0^* + V(1) - V(0)]^+ \Big\} \\ V(n) &= \frac{1}{\gamma+v}\Big\{ [\mu V(n-1) + (v-\mu)V(n)] + \underline{\lambda}\,[p_n^* + V(n+1) - V(n)]^+ \Big\} \\ &\qquad n = 1, 2, \ldots. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} [p_n^* + V(n+1) - V(n)]^+ =&\ \max\Big\{ p_n^* + V(n+1) - V(n), 0 \Big\} \\ =&\ -V(n) + \max\Big\{ p_n^* + V(n+1), V(n) \Big\}. \end{aligned}$$

The Bellman equations are then simplified as

$$\begin{aligned} V(0) &= \frac{\mu + \bar{\lambda} - \underline{\lambda}}{\gamma+v} V(0) + \frac{\underline{\lambda}}{\gamma+v} \max\Big\{ p_0^* + V(1), V(0) \Big\} \\ V(n) &= \frac{\mu}{\gamma+v} V(n-1) + \frac{\bar{\lambda} - \underline{\lambda}}{\gamma+v} V(n) + \frac{\underline{\lambda}}{\gamma+v} \max\Big\{ p_n^* + V(n+1), V(n) \Big\} \\ &\qquad n = 1, 2, \ldots. \end{aligned} \tag{3.5}$$

Recall that in Chen and Frank [5], the arrival rate is a known constant. In our model if we let $\underline{\lambda} = \bar{\lambda}$, the Bellman equations in (3.5) reduce to the equations in their model.

Next, we recall Lemma 1 in Chen and Frank [5], before we introduce Proposition 3.3.2.

Lemma 3.3.1. *Let p_n^* be define in (3.4), let $g(n) = \max\{p_n^* + f(n+1), f(n)\}$. If $p_n^* + f(n+1) - f(n)$ is non-increasing in n, so is $p_n^* + g(n+1) - g(n)$.*

Proof. The proof is in Chen and Frank [5]. □

Proposition 3.3.2. *Let $V(\cdot)$ be the optimal value function satisfying (3.5). Then*

a) $V(\cdot)$ is non-increasing,

b) $p_n^* + V(n+1) - V(n)$ is non-increasing in n, where p_n^* is defined by (3.4),

c) Suppose that $p_0^* \geq 0$, then $p_0^* + V(1) - V(0) \geq 0$.

Proof. We prove this by induction. Let

$$\begin{aligned} V_{m+1}(0) &= \frac{\mu + \bar{\lambda} - \underline{\lambda}}{\gamma + v} V_m(0) + \frac{\underline{\lambda}}{\gamma + v} \max\left\{p_0^* + V_m(1), V_m(0)\right\} \\ V_{m+1}(n) &= \frac{\mu}{\gamma + v} V_m(n-1) + \frac{\bar{\lambda} - \underline{\lambda}}{\gamma + v} V_m(n) \\ &\quad + \frac{\underline{\lambda}}{\gamma + v} \max\left\{p_n^* + V_m(n+1), V_m(n)\right\}. \end{aligned}$$

then $V_m \to V(n)$ as $m \to \infty$.

Consider

$$\begin{aligned} \Delta_{1,m}(n) = \; & [p_n^* + V_m(n+1) - V_m(n)] \\ & - \left[p_{n+1}^* + V_m(n+2) - V_m(n+1)\right] \geq 0, \end{aligned} \tag{3.6}$$

$$\Delta_{2,m}(n) = V_m(n) - V_m(n+1) \geq 0, \tag{3.7}$$

$$\Delta_{3,m}(0) = p_0^* + V_m(1) - V_m(0) \geq 0. \tag{3.8}$$

When $m = 0$, let $V_0(n) = 0$ for all $n \geq 0$, then (3.6)-(3.8) hold. Next suppose (3.6)-(3.8) hold for $m \leq i$, we want to show that these inequalities hold for $m = i + 1$.

We first establish (3.6). Let

$$g_m(n) = \max\Big\{p_n^* + V_m(n+1), V_m(n)\Big\},$$

then

$$\begin{aligned} V_{i+1}(n+1) - V_{i+1}(n) = {} & \frac{\mu}{\gamma+v}\left[V_i(n) - V_i(n-1)\right] + \frac{\bar{\lambda}-\underline{\lambda}}{\gamma+v}\left[V_i(n+1) - V_i(n)\right] \\ & + \frac{\underline{\lambda}}{\gamma+v}\left[g_i(n+1) - g_i(n)\right]. \end{aligned}$$

$$\begin{aligned}
& \Delta_{1,i+1}(n) \\
= {} & \left[p_n^* + V_{i+1}(n+1) - V_{i+1}(n)\right] - \Big[p_{n+1}^* + V_{i+1}(n+2) - V_{i+1}(n+1)\Big] \\
= {} & p_n^* + \frac{\mu}{\gamma+v}\left[V_i(n) - V_i(n-1)\right] + \frac{\bar{\lambda}-\underline{\lambda}}{\gamma+v}\left[V_i(n+1) - V_i(n)\right] \\
& + \frac{\underline{\lambda}}{\gamma+v}\left[g_i(n+1) - g_i(n)\right] - \Bigg[p_{n+1}^* + \frac{\mu}{\gamma+v}\left[V_i(n+1) - V_i(n)\right] \\
& + \frac{\bar{\lambda}-\underline{\lambda}}{\gamma+v}\left[V_i(n+2) - V_i(n+1)\right] + \frac{\underline{\lambda}}{\gamma+v}\left[g_i(n+2) - g_i(n+1)\right]\Bigg] \\
= {} & \frac{\underline{\lambda}}{\gamma+v}\Big\{\left[p_n^* + g_i(n+1) - g_i(n)\right] - \Big[p_{n+1}^* + g_i(n+2) - g_i(n+1)\Big]\Big\} \\
& + \frac{\mu}{\gamma+v}\Big\{\Big[p_{n-1}^* + V_i(n) - V_i(n-1)\Big] - \left[p_n^* + V_i(n+1) - V_i(n)\right]\Big\} \\
& + \frac{\bar{\lambda}-\underline{\lambda}}{\gamma+v}\Big\{\left[p_n^* + V_i(n+1) - V_i(n)\right] - \Big[p_{n+1}^* + V_i(n+2) - V_i(n+1)\Big]\Big\} \\
& + p_n^* - p_{n+1}^* - \frac{\underline{\lambda}}{\gamma+v}(p_n^* - p_{n+1}^*) - \frac{\mu}{\gamma+v}(p_{n-1}^* - p_n^*) - \frac{\bar{\lambda}-\underline{\lambda}}{\gamma+v}(p_n^* - p_{n+1}^*) \\
= {} & \frac{\underline{\lambda}}{\gamma+v}\Big\{\left[p_n^* + g_i(n+1) - g_i(n)\right] - \Big[p_{n+1}^* + g_i(n+2) - g_i(n+1)\Big]\Big\} \\
& + \frac{\mu}{\gamma+v}\Delta_{1,i}(n-1) + \frac{\bar{\lambda}-\underline{\lambda}}{\gamma+v}\Delta_{1,i}(n) \\
& + \frac{\gamma+2\mu}{\gamma+v}p_n^* - \frac{\gamma+\mu}{\gamma+v}p_{n+1}^* - \frac{\mu}{\gamma+v}p_{n-1}^*.
\end{aligned}$$

Since

$$p_n^* = \phi^{n+1}\left(R - \frac{C}{\gamma}\right) - \frac{C}{\gamma}$$
$$\phi = \frac{\mu}{\mu+\gamma}.$$

Then

$$\begin{aligned}
&\frac{\gamma+2\mu}{\gamma+v}p_n^* - \frac{\gamma+\mu}{\gamma+v}p_{n+1}^* - \frac{\mu}{\gamma+v}p_{n-1}^* \\
&= \frac{1}{\bar{\lambda}+\mu+\gamma}\left[-\mu p_{n-1} + (\gamma+2\mu)p_n - (\gamma+\mu)p_{n+1}\right] \\
&= \frac{1}{\bar{\lambda}+\mu+\gamma}\Bigg\{ -\mu\phi^n\left(R-\frac{C}{\gamma}\right) + (2\mu+\gamma)\phi^{n+1}\left(R-\frac{C}{\gamma}\right) \\
&\qquad - (\mu+\gamma)\phi^{n+2}\left(R-\frac{C}{\gamma}\right) + \frac{\mu C}{\gamma} - \frac{(2\mu+\gamma)C}{\gamma} + \frac{(\mu+\gamma)C}{\gamma}\Bigg\} \\
&= \frac{1}{\bar{\lambda}+\mu+\gamma}\Bigg\{\phi^n\left(R-\frac{C}{\gamma}\right)\left[-\mu + (2\mu+\gamma)\frac{\mu}{\mu+\gamma} - \frac{\mu^2}{\mu+\gamma}\right]\Bigg\} \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\Delta_{1,i+1}(n) = {}& \frac{\underline{\lambda}}{\gamma+v}\Big\{ \left[p_n^* + g_i(n+1) - g_i(n)\right] - \big[p_{n+1}^* + g_i(n+2) - g_i(n+1)\big] \Big\} \\
&+ \frac{\mu}{\gamma+v}\Delta_{1,i}(n-1) + \frac{\bar{\lambda}-\underline{\lambda}}{\gamma+v}\Delta_{1,i}(n).
\end{aligned}$$

According to Lemma 3.3.1, since (3.6) holds for $m \le i$, then

$$\left[p_n^* + V_i(n+1) - V_i(n)\right] - \big[p_{n+1}^* + V_i(n+2) - V_i(n+1)\big] \ge 0$$

$$\left[p_n^* + g_i(n+1) - g_i(n)\right] - \big[p_{n+1}^* + g_i(n+2) - g_i(n+1)\big] \ge 0.$$

Then

$$\Delta_{1,i+1}(n) \ge 0.$$

We next show (3.7). According to (3.5),

$$\begin{aligned}\Delta_{2,i+1}(n) &= V_{i+1}(n) - V_{i+1}(n+1) \\ &= \frac{\mu}{\gamma+v}\left[V_i(n-1) - V_i(n)\right] + \frac{\bar{\lambda}-\underline{\lambda}}{\gamma+v}\left[V_i(n) - V_i(n+1)\right] \\ &\quad + \frac{\underline{\lambda}}{\gamma+v}\left[g_i(n) - g_i(n+1)\right] \\ &= \frac{\mu}{\gamma+v}\Delta_{2,i}(n-1) + \frac{\bar{\lambda}-\underline{\lambda}}{\gamma+v}\Delta_{2,i}(n) + \frac{\underline{\lambda}}{\gamma+v}\left[g_i(n) - g_i(n+1)\right].\end{aligned}$$

Since we suppose (3.7) holds for $m \leq i$, then

$$\Delta_{2,i}(n-1) \geq 0, \quad \Delta_{2,i}(n) \geq 0.$$

According to Lemma 3.3.1,

$$g_i(n) - g_i(n+1) \geq 0.$$

So

$$\Delta_{2,i+1}(n) \geq 0.$$

We now establish (3.8). Since

$$\begin{aligned}V_{i+1}(0) &= \frac{\mu+\bar{\lambda}-\underline{\lambda}}{\gamma+v}V_i(0) + \frac{\underline{\lambda}}{\gamma+v}\max\left\{p_0^* + V_i(1), V_i(0)\right\} \\ V_{i+1}(1) &= \frac{\mu}{\gamma+v}V_i(0) + \frac{\bar{\lambda}-\underline{\lambda}}{\gamma+v}V_i(1) + \frac{\underline{\lambda}}{\gamma+v}\max\left\{p_1^* + V_i(2), V_i(1)\right\}.\end{aligned}$$

Then

$$\begin{aligned}
\Delta_{3,i+1}(0) &= p_0^* + V_{i+1}(1) - V_{i+1}(0) \\
&= p_0^* + \frac{\mu}{\gamma+v} V_i(0) + \frac{\bar{\lambda}-\underline{\lambda}}{\gamma+v} V_i(1) + \frac{\underline{\lambda}}{\gamma+v} \max\left\{p_1^* + V_i(2), V_i(1)\right\} \\
&\quad - \frac{\mu+\bar{\lambda}-\underline{\lambda}}{\gamma+v} V_i(0) + \frac{\underline{\lambda}}{\gamma+v} \max\left\{p_0^* + V_i(1), V_i(0)\right\} \\
&= p_0^* - \frac{\bar{\lambda}-\underline{\lambda}}{\gamma+v} \left[V_i(0) - V_i(1)\right] \\
&\quad + \frac{\underline{\lambda}}{\gamma+v} \Big\{ \max\left\{p_1^* + V_i(2), V_i(1)\right\} - \max\left\{p_0^* + V_i(1), V_i(0)\right\}\Big\}.
\end{aligned}$$

Since we suppose (3.8) holds for $m \leq i$, then

$$p_0^* + V_i(1) - V_i(0) \geq 0.$$

Then

$$\begin{aligned}
\Delta_{3,i+1}(0) &= p_0^* - \frac{\bar{\lambda}-\underline{\lambda}}{\gamma+v} \left[V_i(0) - V_i(1)\right] + \frac{\underline{\lambda}}{\gamma+v} \Big\{ \max\left\{p_1^* + V_i(2), V_i(1)\right\} \\
&\quad - \left[p_0^* + V_i(1)\right] \Big\} \\
&= p_0^* + \frac{\bar{\lambda}-\underline{\lambda}}{\gamma+v} \left[V_i(1) - V_i(0)\right] + \frac{\underline{\lambda}}{\gamma+v} \left[p_1^* + V_i(2) - V_i(1)\right]^+ - \frac{\underline{\lambda}}{\gamma+v} p_0^* \\
&= \frac{\bar{\lambda}-\underline{\lambda}}{\gamma+v} \left[p_0^* + V_i(1) - V_i(0)\right] + \frac{\gamma+\mu}{\gamma+v} p_0^* + \frac{\underline{\lambda}}{\gamma+v} \left[p_1^* + V_i(2) - V_i(1)\right]^+ .
\end{aligned}$$

Given $p_0^* \geq 0$, $\Delta_{3,i+1}(0) \geq 0$. $\square$

Next, we find the optimal policy for the RM based on Proposition 3.3.2.

Let

$$n_R = \sup\{n : p_n^* + V(n+1) - V(n) \geq 0\}.$$

If $p_0^* \geq 0$ then $n_R \geq 0$. Furthermore,

$$p_n^* + V(n+1) - V(n) = \begin{cases} \geq 0 & n \leq n_R \\ < 0 & n > n_R. \end{cases}$$

When $n \leq n_R$, the RM charges a fee of p_n^* to encourage new customers to enter the system. When $n > n_R$, it is not optimal for the RM to charge the fee p_n^* since $p_n^* + V(n+1) < V(n)$, so the RM charges a fee that is larger than p_n^*, denoted by $(p_n^*)^+$. According to the definition of p_n^* in (3.4), an arriving customer does not enter the system.

For the optimal policy of nature, when $p_n^* + V(n+1) - V(n) \geq 0$, or equivalently, $n \leq n_R$, nature chooses the value of $\underline{\lambda}$. When $p_n^* + V(n+1) - V(n) < 0$, nature chooses any value from the set $[\underline{\lambda}, \bar{\lambda}]$. Thus an optimal policy for nature is to choose $\Lambda_t = \underline{\lambda}$ at each decision epoch.

Thus for the original static adversarial model in which nature chooses a fixed arrival rate from the beginning of the system, an optimal policy for the nature is to choose $\underline{\lambda}$.

3.4 Robust Social Benefit Rate

In this section we consider the SO's problem with the objective of maximizing the total discounted social benefit rate of the entire society. Assuming a customer who does not enter receives zero benefit, suppose a new customer arrives when the queue length is n. Then, the expected social benefit from the new customer is

$$
\begin{aligned}
e^{-\gamma\tau}R - C\int_{u=0}^{\tau} e^{-\gamma u}du &= e^{-\gamma\tau}R - \frac{C}{\gamma}\left(1-e^{-\gamma\tau}\right)\\
&= \phi^{n+1}\left(R+\frac{C}{\gamma}\right) - \frac{C}{\gamma}\\
&= p_n^*.
\end{aligned}
$$

Thus the Bellman equations for the social optimization are the same as revenue maximization. According to (3.5), the social-optimization problem and the revenue-maximizing problem have the same optimal threshold. An arriving customer enters the system when the queue length $n \leq n_R$, and balks when $n > n_R$.

3.5 Computational Study

In this section, we study how the lower and upper bounds of the random arrival rate affect the optimal threshold and social benefit rate.

We give two computational examples. In the first case we change the lower bound of the random arrival rate, while in the second case we change the upper bound. The parameters are shown in Table 3.1.

Homogeneous Customers	$\underline{\lambda}$	$\bar{\lambda}$	μ	R	C	γ
Case 1	[0.01,20]	20	1	100	10	0.095
Case 2	0.5	[0.5,10]	1	100	10	0.095

Table 3.1: Homogeneous Customers, Optimal Threshold

For the first case, figure 3.1 presents the change of optimal threshold to the lower bound value. Figure 3.2 shows the change of the optimal social benefit rate to the lower bound value. We observe that the optimal threshold is a non-increasing function of the lower bound, while the social benefit rate is a non-decreasing function. For the second case, figure 3.3 presents the change of the optimal threshold to the upper bound value, and as we expected, the optimal threshold does not change with it. So in this model, only the lower bound affects the optimal threshold and the social benefit rate.

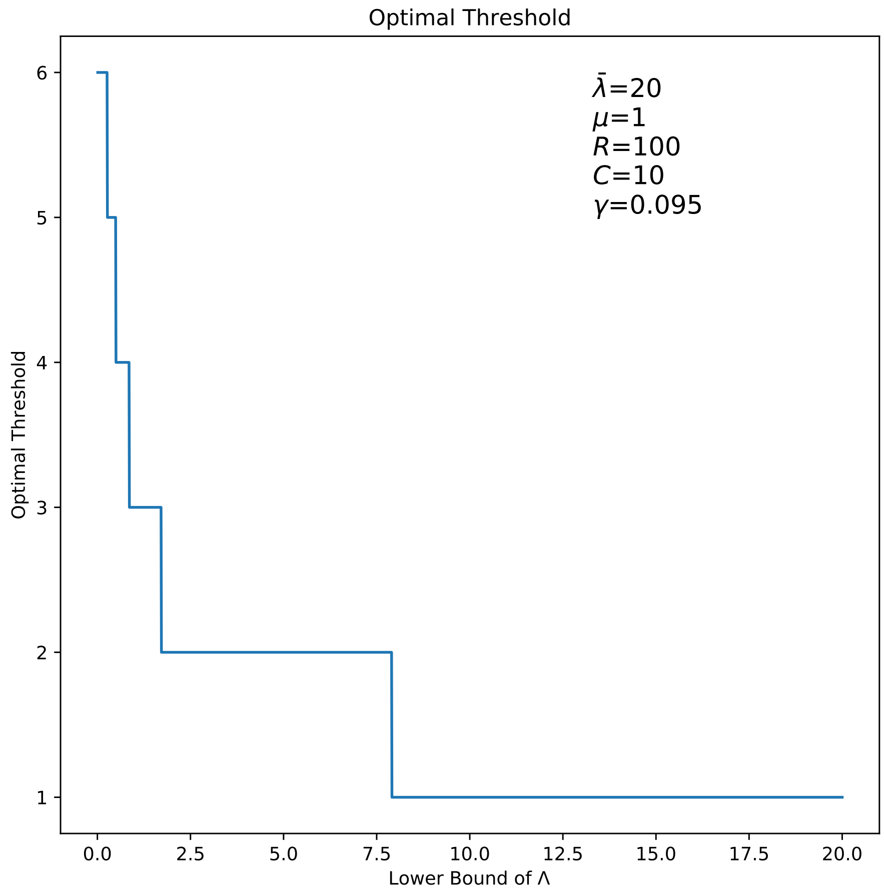

Figure 3.1: Homogeneous Customers, case 1, Optimal Threshold

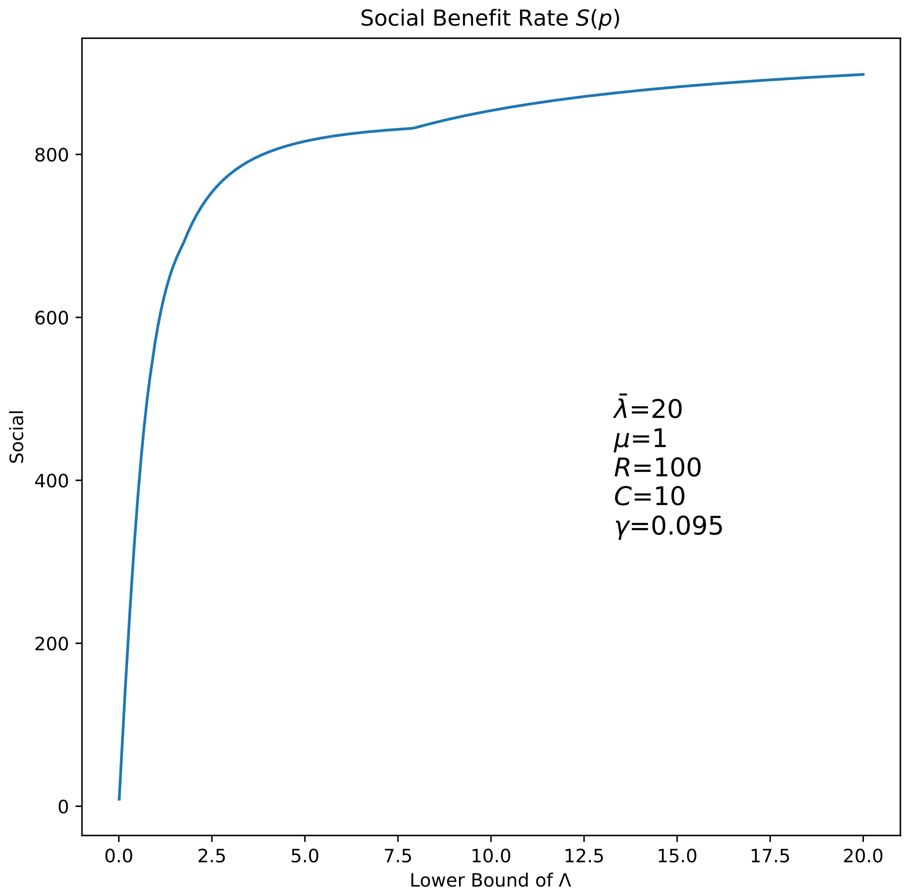

Figure 3.2: Homogeneous Customers, case 1, Social Benefit Rate

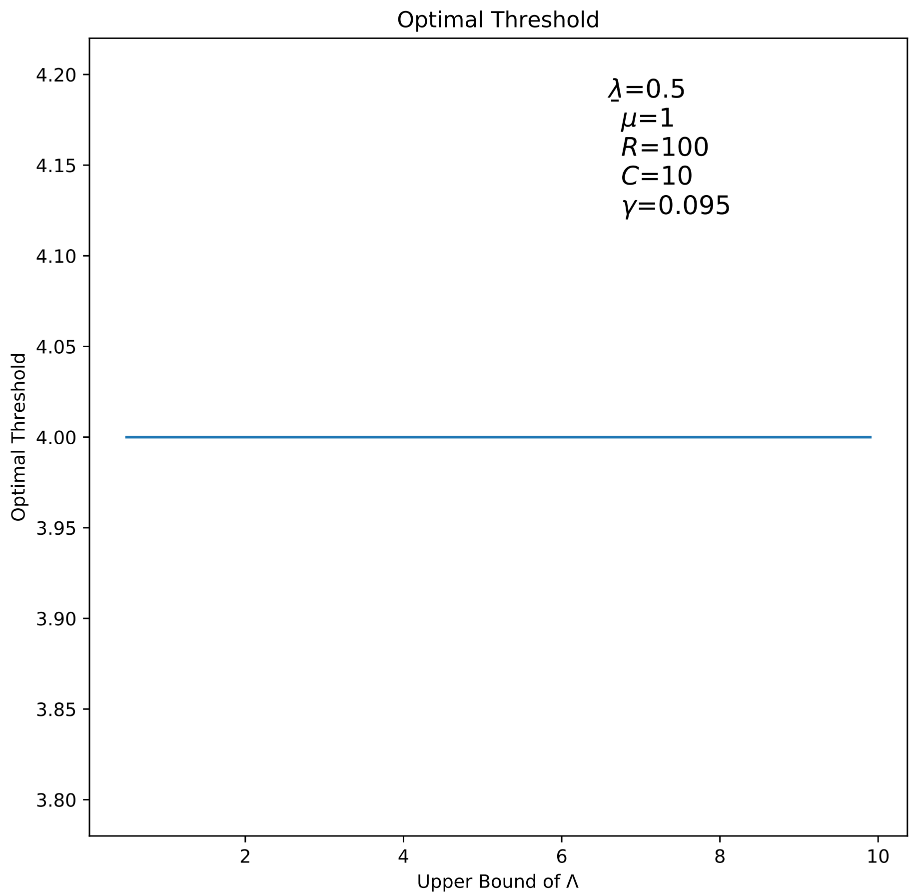

Figure 3.3: Homogeneous Customers, case 2, Optimal Threshold

Chapter 4

State-dependent Pricing with Uncertain Arrival Rate and Heterogeneous Customers

In this chapter, we follow the same model assumptions in Chapter 3 except for the ones regarding customer characteristics. Instead of homogeneous customers, we consider heterogeneous customers. The service value and the waiting cost rate are assumed to be random variables and are denoted by X and Y, respectively. We set $E(X) = R$, $E(Y) = C$. The system manager does not observe the values of individual customers, but only knows the joint cdf of X and Y.

4.1 Individual Customer Decision

Suppose there is a new customer with service value x and waiting cost rate y. He observes the queue length n and the admission price p_n upon arrival. Let τ denote his waiting time in the system. The customer enters the system if and only if his net benefit is non-negative:

$$
\begin{aligned}
E\left[e^{-\gamma\tau}x - p_n - y\int_0^{\tau} e^{-\gamma t}dt\right] &= \phi^{n+1}\left(x + \frac{y}{\gamma}\right) - \left(p_n + \frac{y}{\gamma}\right) \\
&\geq 0, \qquad \text{where } \phi = \frac{\mu}{\mu+\gamma}.
\end{aligned}
$$

Given queue length n, the fraction of customer population who enter the system is given by

$$P_{X,Y}\left(\phi^{n+1}X - \frac{1-\phi^{n+1}}{\gamma}Y \geq p_n\right).$$

Now define the random variables

$$\Theta_n = \phi^n X - \frac{1-\phi^n}{\gamma}Y, \quad \forall n \geq 1, \forall p_n \geq 0. \tag{4.1}$$

Then $\bar{F}_{\Theta_{n+1}}(p_n)$ is the joining fraction and

$$\bar{F}_{\Theta_{n+1}}(p_n) = P_{X,Y}\left(\phi^{n+1}X - \frac{1-\phi^{n+1}}{\gamma}Y \geq p_n\right). \tag{4.2}$$

Lemma 4.1.1. *$\bar{F}_{\Theta_{n+1}}(p_n)$ is non-increasing in both n and p_n.*

Proof. First, it is trivial that $\forall n \geq 0$, $\bar{F}_{\Theta_{n+1}}(p_n)$ is non-increasing in p_n. When the value of admission fee is fixed, if let

$$\vec{a} = \begin{pmatrix} \phi^n \\ \frac{1-\phi^n}{\gamma} \end{pmatrix} \qquad \vec{Z} = \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Then

$$\Theta_n = \vec{a} \cdot \vec{Z}.$$

Since $||\vec{a}||$ is non-decreasing in p, $\bar{F}^{\Theta}\left(\hat{\theta}(p,n)\right)$ is non-increasing in both n and p. $\square$

Given admission fee p_n, according to (4.1), the transition probabilities are

$$
\begin{aligned}
q_{n,n-1} &= \frac{\mu}{v} \quad n > 0 \\
q_{n,n+1} &= \begin{cases} \frac{\Lambda}{v} & \Theta_{n+1} > p_n, n \geq 0 \\ 0 & \Theta_{n+1} \leq p_n, n \geq 0 \end{cases} \\
q_{nn} &= \begin{cases} \frac{v-\Lambda}{v} & \Theta_{n+1} > p_n, n = 0 \\ 1 & \Theta_{n+1} \leq p_n, n = 0 \\ \frac{v-\Lambda-\mu}{v} & \Theta_{n+1} > p_n, n > 0 \\ \frac{v-\mu}{v} & \Theta_{n+1} \leq p_n, n > 0. \end{cases}
\end{aligned}
$$

4.2 Robust Revenue Rate

In this section we derive the Bellman equations to solve for the optimal pricing policy of the RM. Let $V(n)$ be the expected revenue rate starting from queue length n, $n \geq 0$. The robust Bellman equations are formulated as:

$$
\begin{aligned}
V(0) &= \frac{1}{\gamma+v} \max_{p_0} \min_{\Lambda} \Big\{ \Lambda \bar{F}_{\Theta_1}(p_0) \left[p_0 + V(1)\right] + (v-\Lambda)\bar{F}_{\Theta_1}(p_0)V(0) \\
&\quad + vF_{\Theta_1}(p_0)V(0) \Big\} \\
&= \frac{1}{\gamma+v} \max_{p_0} \min_{\Lambda} \Big\{ vV(0) + \Lambda \bar{F}_{\Theta_1}(p_0) \left[p_0 + V(1) - V(0)\right] \Big\} \\
&= \frac{v}{\gamma+v} V(0) + \frac{1}{\gamma+v} \max_{p_0} \min_{\Lambda} \Big\{ \Lambda \bar{F}_{\Theta_1}(p_0) \left[p_0 + V(1) - V(0)\right] \Big\} \\
V(n) &= \frac{1}{\gamma+v} \max_{p_n} \min_{\Lambda} \Big\{ \Lambda \bar{F}_{\Theta_{n+1}}(p_n) \left[p_n + V(n+1)\right] + \mu V(n-1) \\
&\quad + (v-\mu)F_{\Theta_{n+1}}(p_n)V(n) + (v-\Lambda-\mu)\bar{F}_{\Theta_{n+1}}(p_n)V(n) \Big\} \\
&= \frac{1}{\gamma+v} \max_{p_n} \min_{\Lambda} \Big\{ (v-\mu)V(n) + \mu V(n-1) \\
&\quad + \Lambda \bar{F}_{\Theta_{n+1}}(p_n) \left[p_n + V(n+1) - V(n)\right] \Big\} \\
&= \frac{\bar{\lambda}}{\gamma+v} V(n) + \frac{\mu}{\gamma+v} V(n-1) \\
&\quad + \frac{1}{\gamma+v} \max_{p_n} \min_{\Lambda} \Big\{ \Lambda \bar{F}_{\Theta_{n+1}}(p_n) \left[p_n + V(n+1) - V(n)\right] \Big\} \\
&\quad n = 1, 2, \ldots.
\end{aligned} \tag{4.3}
$$

These equations are then simplified as:

$$
\begin{aligned}
V(0) &= \frac{v}{\gamma+v} V(0) + \frac{\underline{\lambda}}{\gamma+v} \max_{p_0} \bar{F}_{\Theta_1}(p_0) \left[p_0 + V(1) - V(0)\right] \\
V(n) &= \frac{\mu}{\gamma+v} V(n-1) + \frac{\bar{\lambda}}{\gamma+v} V(n) \\
&\quad + \frac{\underline{\lambda}}{\gamma+v} \max_{p_n} \bar{F}_{\Theta_{n+1}}(p_n) \left[p_n + V(n+1) - V(n)\right] \\
&\quad n = 1, 2, \ldots.
\end{aligned} \tag{4.4}
$$

Furthermore,

$$\begin{aligned}
V(0) &= \frac{\mu}{\gamma+v}V(0) + \frac{\bar{\lambda}}{\gamma+v}\max_{p_0}\Big\{\left[1-\frac{\lambda}{\bar{\bar{\lambda}}}\bar{F}_{\Theta_1}(p_0)\right]V(0) \\
&\quad + \frac{\lambda}{\bar{\bar{\lambda}}}\bar{F}_{\Theta_1}(p_0)\left[p_0+V(1)\right]\Big\} \\
V(n) &= \frac{\mu}{\gamma+v}V(n-1) + \frac{\bar{\lambda}}{\gamma+v}\max_{p_n}\Big\{\left[1-\frac{\lambda}{\bar{\bar{\lambda}}}\bar{F}_{\Theta_{n+1}}(p_n)\right]V(n) \\
&\quad + \frac{\lambda}{\bar{\bar{\lambda}}}\bar{F}_{\Theta_{n+1}}(p_n)\left[p_n+V(n+1)\right]\Big\} \\
&\quad n = 1,2,\ldots.
\end{aligned} \tag{4.5}$$

Lemma 4.2.1. *Let $S = 0,1,2,\ldots$, $V : S \to \mathcal{R}$, $u : S \to \mathcal{R}^+$. Consider the following mappings:*

$$(TV)(n) = \begin{cases} \frac{\mu}{\gamma+v}V(0) + \frac{\bar{\lambda}}{\gamma+v}\max_{p_0}\Big\{\left[1-\frac{\lambda}{\bar{\bar{\lambda}}}\bar{F}_{\Theta_1}(p_0)\right]V(0) \\ \qquad + \frac{\lambda}{\bar{\bar{\lambda}}}\bar{F}_{\Theta_1}(p_0)\left[p_0+V(1)\right]\Big\} & n=0 \\ \frac{\mu}{\gamma+v}V(n-1) + \frac{\bar{\lambda}}{\gamma+v}\max_{p_n}\Big\{\left[1-\frac{\lambda}{\bar{\bar{\lambda}}}\bar{F}_{\Theta_{n+1}}(p_n)\right]V(n) \\ \qquad + \frac{\lambda}{\bar{\bar{\lambda}}}\bar{F}_{\Theta_{n+1}}(p_n)\left[p_n+V(n+1)\right]\Big\} & n=1,2,\ldots. \end{cases}$$

$$(T_\mu V)(n) = \begin{cases} \frac{\mu}{\gamma+v}V(0) + \frac{\bar{\lambda}}{\gamma+v}\max_{p_0^u}\Big\{\left[1-\frac{\lambda}{\bar{\bar{\lambda}}}\bar{F}_{\Theta_1}(p_0^u)\right]V(0) \\ \qquad + \frac{\lambda}{\bar{\bar{\lambda}}}\bar{F}_{\Theta_1}(p_0^u)\left[p_0^u+V(1)\right]\Big\} & n=0 \\ \frac{\mu}{\gamma+v}V(n-1) + \frac{\bar{\lambda}}{\gamma+v}\max_{p_n^u}\Big\{\left[1-\frac{\lambda}{\bar{\bar{\lambda}}}\bar{F}_{\Theta_{n+1}}(p_n^u)\right]V(n) \\ \qquad + \frac{\lambda}{\bar{\bar{\lambda}}}\bar{F}_{\Theta_{n+1}}(p_n^u)\left[p_n^u+V(n+1)\right]\Big\} & n=1,2,\ldots. \end{cases}$$

Then

1. *If $V(n) \geq \hat{V}(n)$, then $(T^kV)(n) \geq (T^k\hat{V})(n)$, $(T_u^kV)(n) \geq (T_u^k\hat{V})(n)$ for $k = 1,2,\ldots$.*

2. *Let* $e : S \to \mathcal{R}$ *s.t.* $e(x) = 1$, $\forall x \in S$. *Then* $(T^k(V + r \cdot e))(n) = (T^k V)(n) + \alpha^k \cdot r$, $(T_u^k(V + r \cdot e))(n) = (T_u^k V)(n) + \alpha^k \cdot r$, *where* $\alpha = \frac{v}{\gamma + v}$.

3. T *is a contraction mapping.*

Proof. Let

$$\begin{aligned} p_n^* &= \underset{p_n}{\operatorname{argmax}} \left\{ \left[1 - \frac{\lambda}{\bar{\bar{\lambda}}} \bar{F}_{\Theta_{n+1}}(p_n) \right] V(n) + \frac{\lambda}{\bar{\bar{\lambda}}} \bar{F}_{\Theta_{n+1}}(p_n) \left[p_n + V(n+1) \right] \right\} \\ \hat{p}_n^* &= \underset{p_n}{\operatorname{argmax}} \left\{ \left[1 - \frac{\lambda}{\bar{\bar{\lambda}}} \bar{F}_{\Theta_{n+1}}(p_n) \right] \hat{V}(n) + \frac{\lambda}{\bar{\bar{\lambda}}} \bar{F}_{\Theta_{n+1}}(p_n) \left[p_n + \hat{V}(n+1) \right] \right\} \\ & \qquad n = 0, 1, 2, \ldots . \end{aligned}$$

$$
\begin{aligned}
TV(n) &= \begin{cases} \dfrac{\mu}{\gamma+v}V(0) + \dfrac{\bar{\lambda}}{\gamma+v}\max\limits_{p_0}\Big\{ \left[1-\dfrac{\underline{\lambda}}{\bar{\lambda}}\bar{F}_{\Theta_1}(p_0)\right]V(0) \\ \qquad\qquad\qquad\qquad\qquad + \dfrac{\underline{\lambda}}{\bar{\lambda}}\bar{F}_{\Theta_1}(p_0)\left[p_0+V(1)\right]\Big\} \\ \dfrac{\mu}{\gamma+v}V(n-1) + \dfrac{\bar{\lambda}}{\gamma+v}\max\limits_{p_n}\Big\{ \left[1-\dfrac{\underline{\lambda}}{\bar{\lambda}}\bar{F}_{\Theta_{n+1}}(p_n)\right]V(n) \\ \qquad\qquad\qquad\qquad\qquad + \dfrac{\underline{\lambda}}{\bar{\lambda}}\bar{F}_{\Theta_{n+1}}(p_n)\left[p_n+V(n+1)\right]\Big\} \end{cases} \\
&= \begin{cases} \frac{v}{\gamma+v}\max_{p_0}\left\{\frac{\underline{\lambda}\bar{F}_{\Theta_1}(p_0)}{v}\left[p_0+V(1)\right] + \frac{v-\underline{\lambda}\bar{F}_{\Theta_1}(p_0)}{v}V(0)\right\} \\ \dfrac{v}{\gamma+v}\max\limits_{p_n}\Big\{\dfrac{\underline{\lambda}\bar{F}_{\Theta_{n+1}}(p_n)}{v}\left[p_n+V(n+1)\right] + \dfrac{\mu}{v}V(n-1) \\ \qquad\qquad\qquad\qquad\qquad + \dfrac{v-\mu-\underline{\lambda}\bar{F}_{\Theta_{n+1}}(p_n)}{v}V(n)\Big\} \end{cases} \\
&= \begin{cases} \frac{v}{\gamma+v}\left\{\frac{\underline{\lambda}\bar{F}_{\Theta_1}(p_0^*)}{v}\left[p_0^*+V(1)\right] + \frac{v-\underline{\lambda}\bar{F}_{\Theta_1}(p_0^*)}{v}V(0)\right\} \\ \dfrac{v}{\gamma+v}\Big\{\dfrac{\underline{\lambda}\bar{F}_{\Theta_{n+1}}(p_n^*)}{v}\left[p_n^*+V(n+1)\right] + \dfrac{\mu}{v}V(n-1) \\ \qquad\qquad\qquad\qquad + \dfrac{v-\mu-\underline{\lambda}\bar{F}_{\Theta_{n+1}}(p_n^*)}{v}V(n)\Big\} \end{cases} \\
&\geq \begin{cases} \frac{v}{\gamma+v}\left\{\frac{\underline{\lambda}\bar{F}_{\Theta_1}(\hat{p}_0^*)}{v}\left[\hat{p}_0^*+V(1)\right] + \frac{v-\underline{\lambda}\bar{F}_{\Theta_1}(\hat{p}_0^*)}{v}V(0)\right\} \\ \dfrac{v}{\gamma+v}\Big\{\dfrac{\underline{\lambda}\bar{F}_{\Theta_{n+1}}(\hat{p}_n^*)}{v}\left[\hat{p}_n^*+V(n+1)\right] + \dfrac{\mu}{v}V(n-1) \\ \qquad\qquad\qquad\qquad + \dfrac{v-\mu-\underline{\lambda}\bar{F}_{\Theta_{n+1}}(\hat{p}_n^*)}{v}V(n)\Big\} \end{cases} \\
&\geq \begin{cases} \frac{v}{\gamma+v}\left\{\frac{\underline{\lambda}\bar{F}_{\Theta_1}(\hat{p}_0^*)}{v}\left[\hat{p}_0^*+\hat{V}(1)\right] + \frac{v-\underline{\lambda}\bar{F}_{\Theta_1}(\hat{p}_0^*)}{v}\hat{V}(0)\right\} & n=0 \\ \dfrac{v}{\gamma+v}\Big\{\dfrac{\underline{\lambda}\bar{F}_{\Theta_{n+1}}(\hat{p}_n^*)}{v}\left[\hat{p}_n^*+\hat{V}(n+1)\right] + \dfrac{\mu}{v}\hat{V}(n-1) & \\ \qquad\qquad\qquad\qquad + \dfrac{v-\mu-\underline{\lambda}\bar{F}_{\Theta_{n+1}}(\hat{p}_n^*)}{v}\hat{V}(n)\Big\} & n=1,2,\ldots. \end{cases} \\
&= T\hat{V}(n).
\end{aligned}
$$

Then $(T^kV)(n) \geq (T^k\hat{V})(n)$ and $(T_u^kV)(n) \geq (T_u^k\hat{V})(n)$ for $k = 1, 2, \ldots.$

$$T(V+r\cdot e)(n)=\begin{cases}\dfrac{v}{\gamma+v}\max\limits_{p_0}\Big\{\dfrac{\underline{\lambda}\bar{F}_{\Theta_1}(p_0)}{v}\left[p_0+(V+r\cdot e)(1)\right]\\ \qquad\qquad +\dfrac{v-\underline{\lambda}\bar{F}_{\Theta_1}(p_0)}{v}(V+r\cdot e)(0)\Big\}\\ \dfrac{v}{\gamma+v}\max\limits_{p_n}\Big\{\dfrac{\underline{\lambda}\bar{F}_{\Theta_{n+1}}(p_n)}{v}\left[p_n+(V+r\cdot e)(n+1)\right]\\ \qquad\qquad +\dfrac{\mu}{v}(V+r\cdot e)(n-1)\\ \qquad\qquad +\dfrac{v-\mu-\underline{\lambda}\bar{F}_{\Theta_{n+1}}(p_n)}{v}(V+r\cdot e)(n)\Big\}\end{cases}$$

$$=\begin{cases}\dfrac{v}{\gamma+v}\max\limits_{p_0}\Big\{\dfrac{\underline{\lambda}\bar{F}_{\Theta_1}(p_0)}{v}\left[p_0+V(1)\right] & \\ \quad+\dfrac{v-\underline{\lambda}\bar{F}_{\Theta_1}(p_0)}{v}V(0)\Big\}+\dfrac{v}{\gamma+v}r & n=0\\ \dfrac{v}{\gamma+v}\max\limits_{p_n}\Big\{\dfrac{\underline{\lambda}\bar{F}_{\Theta_{n+1}}(p_n)}{v}\left[p_n+V(n+1)\right] & \\ +\dfrac{\mu}{v}V(n-1)+\dfrac{v-\mu-\underline{\lambda}\bar{F}_{\Theta_{n+1}}(p_n)}{v}V(n)\Big\} & n=1,2,\ldots\\ \qquad +\dfrac{v}{\gamma+v}r & \end{cases}$$

$$=(TV)(n)+\frac{v}{\gamma+v}r.$$

Then $(T^k(V+r\cdot e))(n)=(T^kV)(n)+\alpha^k\cdot r$, $(T_u^k(V+r\cdot e))(n)=(T_u^kV)(n)+\alpha^k\cdot r$, where $\alpha=\frac{v}{\gamma+v}$. □

Theorem 4.2.2. *The Bellman equations derived in (4.4) have a unique fixed point, which is the optimal revenue rate V^*.*

Proof. The proof is similar to the standard proof based on the Banach fixed point theorem. □

4.3 Robust Social Benefit Rate

In this section we examine the total discounted social benefit rate. Suppose a new customer arrives when the queue length is n. Let r_n denote the expected social benefit from this customer, then

$$\begin{aligned} r_n &= E\left[e^{-\gamma\tau}X - Y\int_0^\tau e^{-\gamma t}dt\right] \\ &= \phi^{n+1}\left(R + \frac{C}{\gamma}\right) - \frac{C}{\gamma} \\ & n = 0, 1, \dots. \end{aligned}$$

Let $W^*(n)$ be the optimal robust social benefit rate starting at queue length n, $n \geq 0$. $W^*(n)$ is the unique solution of the following Bellman equations:

$$\begin{aligned} W(0) &= \frac{v}{\gamma+v}W(0) + \frac{\lambda}{\gamma+v}\max_{p_0}\bar{F}_{\Theta_1}(p_0)\left[r_0 + W(1) - W(0)\right] \\ W(n) &= \frac{\mu}{\gamma+v}W(n-1) + \frac{\bar{\lambda}}{\gamma+v}W(n) \\ &\quad + \frac{\lambda}{\gamma+v}\max_{p_n}\bar{F}_{\Theta_{n+1}}(p_n)\left[r_n + W(n+1) - W(n)\right] \\ &\quad n = 1, 2, \dots. \end{aligned} \tag{4.6}$$

Since $r_n + W(n+1) - W(n)$ does not depend on p_n, then

$$\begin{aligned} p_n^* &= \begin{cases} \operatorname{argmax}_{p_n}\bar{F}_{\Theta_{n+1}}(p_n), & r_n + W(n+1) - W(n) \geq 0 \\ \operatorname{argmin}_{p_n}\bar{F}_{\Theta_{n+1}}(p_n), & r_n + W(n+1) - W(n) \leq 0 \end{cases} \\ &= \begin{cases} 0, & r_n + W(n+1) - W(n) \geq 0 \\ \infty, & r_n + W(n+1) - W(n) \leq 0. \end{cases} \quad n = 0, 1, \dots. \end{aligned}$$

The Bellman equations for the social benefit rate are derived as:

$$
\begin{aligned}
W(0) &= \frac{v}{\gamma+v}W(0) + \frac{\underline{\lambda}}{\gamma+v}\bar{F}_{\Theta_1}(0)\left[r_0 + W(1) - W(0)\right]^+ \\
W(n) &= \frac{\mu}{\gamma+v}W(n-1) + \frac{\bar{\lambda}}{\gamma+v}W(n) \\
&\quad + \frac{\underline{\lambda}}{\gamma+v}\bar{F}_{\Theta_{n+1}}(0)\left[r_n + W(n+1) - W(n)\right]^+ \\
&\quad n = 1, 2, \ldots.
\end{aligned} \tag{4.7}
$$

Furthermore, we have

$$
\begin{aligned}
\left[r_n + W(n+1) - W(n)\right]^+ &= \max\left\{r_n + W(n+1) - W(n), 0\right\} \\
&= \max\left\{r_n + W(n+1), W(n)\right\} - W(n).
\end{aligned}
$$

Then

$$
\begin{aligned}
W(0) &= \frac{\mu + \bar{\lambda} - \underline{\lambda}\bar{F}_{\Theta_1}(0)}{\gamma+v}W(0) + \frac{\underline{\lambda}}{\gamma+v}\bar{F}_{\Theta_1}(0)\max\left\{r_0 + W(1), W(0)\right\} \\
W(n) &= \frac{\mu}{\gamma+v}W(n-1) + \frac{\bar{\lambda} - \underline{\lambda}\bar{F}_{\Theta_{n+1}}(0)}{\gamma+v}W(n) \\
&\quad + \frac{\underline{\lambda}}{\gamma+v}\bar{F}_{\Theta_{n+1}}(0)\max\left\{r_n + W(n+1), W(n)\right\} \\
&\quad n = 1, 2, \ldots.
\end{aligned} \tag{4.8}
$$

The Bellman equations for the social benefit rate are similar to the equations we derived in Chapter 3, except that there is a joining fraction at zero fee multiplied by the lower bound of the random arrival rate. So the optimal policy for the SO is of the threshold type. And nature chooses $\underline{\lambda}$ at all times.

4.4 Computational Study

In this section we solve for the optimal pricing policies for the RM and the SO, based on the Bellman equations in (4.5) and (4.8). We utilize the value iteration algorithm with error bounds [3] to find the optimal pricing policy.

We first define two functions of p.

$$q_n(p) = \frac{\lambda}{\bar{\lambda}} \bar{F}_{\Theta_{n+1}}(p) \tag{4.9}$$

$$g_n(p) = (1 - q_n(p)) V(n) + q_n(p) [p + V(n+1)] . \tag{4.10}$$

We use the same mapping defined in Lemma 4.2.1. We now introduce the value iteration algorithm.

Algorithm 1 Value Iteration

Input

State space $S = \{0, 1, \dots\}$
Action space $A = \mathbb{R}^+$
Reward function $r : S \times A \to \mathbb{R}$
Transition probabilities $q_{ij}(a) = P(j|i, a)$
Discounting factor $\alpha \in (0, 1)$
Tolerance $tol > 0$

Output

Optimal admission fee p_i^*, $i \in \mathcal{S}$
Optimal value function $J^*(i)$, $i \in \mathcal{S}$

procedure
  Initialize $J_0(S)$ arbitrarily
  $k \leftarrow 0$
  repeat
    $k \leftarrow k + 1$
    for each state $i \in \mathcal{S}$ **do**
      $J_k(i) \leftarrow TJ_k(i)$
    end for
    $\underline{c}_k = \frac{\alpha}{1-\alpha} \min_{i \in S} [J_k(i) - J_{k-1}(i)]$
    $\bar{c}_k = \frac{\alpha}{1-\alpha} \max_{i \in S} [J_k(i) - J_{k-1}(i)]$
  until $\bar{c}_k - \underline{c}_k < t$
  for each state $i \in \mathcal{S}$ **do**
    $p_i^* \leftarrow \text{argmax}_{p_i}\, g_i^k(p_i)$
    $J^*(i) \leftarrow J_k(i) + \frac{\underline{c}_k + \bar{c}_k}{2} \vec{e}$
  end for
  return
end procedure

Next, we give some computational examples using this algorithm.

4.4.1 Exponential Service Uniform Cost

Assume the service value X and the waiting cost rate Y are independent. In this section we discuss a case when $X \sim \exp(\frac{1}{R})$ and $Y \sim U[C-\epsilon, C+\epsilon]$, where $\epsilon \leq C$. R, C and ϵ are positive constants.

Theorem 4.4.1. *Given X and Y are independent random variables. And $X \sim \exp(\frac{1}{R})$ and $Y \sim U[C-\epsilon, C+\epsilon]$, where $\epsilon \leq C$. R, C and ϵ are positive constants. As defined in (4.10), the function $g_n(p)$ is a unimodal function in terms of p.*

Proof. The joint pdf of X and Y is

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} \frac{1}{2\epsilon R}e^{-\frac{1}{R}x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

The joining fraction of the customer population is

$$\begin{aligned}
\bar{F}_{\Theta_n}(p) &= \Pr_{X,Y}\left(\phi^n X - \frac{1-\phi^n}{\gamma}Y \geq p\right) \\
&= \iint_{\substack{\phi^n x - \frac{1-\phi^n}{\gamma}y \geq p \\ C-\epsilon \leq y \leq C+\epsilon}} f_{X,Y}(x,y)dxdy \\
&= \frac{1}{2\epsilon}\int_{C-\epsilon}^{C+\epsilon}\int_{\frac{1-\phi^n}{\gamma\phi^n}y+\frac{p}{\phi^n}}^{\infty} \frac{1}{R}e^{-\frac{1}{R}x}dxdy \\
&= \frac{1}{2\epsilon}\int_{C-\epsilon}^{C+\epsilon} \exp\left(-\frac{1}{R}\left(\frac{1-\phi^n}{\gamma\phi^n}y + \frac{p}{\phi^n}\right)\right)dy \\
&= \frac{1}{2\epsilon}e^{-\frac{p}{R\phi^n}}\int_{C-\epsilon}^{C+\epsilon} e^{-\frac{1-\phi^n}{\gamma R\phi^n}y}dy \\
&= \frac{\gamma R\phi^n}{2\epsilon\left(1-\phi^n\right)}e^{-\frac{p}{R\phi^n}}\left(e^{-\frac{1-\phi^n}{\gamma R\phi^n}(C-\epsilon)} - e^{-\frac{1-\phi^n}{\gamma R\phi^n}(C+\epsilon)}\right).
\end{aligned} \tag{4.11}$$

According to (4.11), let

$$\bar{F}_{\Theta_n}(p) = a_n e^{-b_n p}.$$

Then

$$\begin{aligned}
a_{n+1} &= \frac{\gamma R \phi^{n+1}}{2\epsilon\left(1-\phi^{n+1}\right)} \left(e^{-\frac{1-\phi^{n+1}}{\gamma R \phi^{n+1}}(C-\epsilon)} - e^{-\frac{1-\phi^{n+1}}{\gamma R \phi^{n+1}}(C+\epsilon)} \right) \\
b_{n+1} &= \frac{1}{R\phi^{n+1}}.
\end{aligned}$$

Using (4.9) we have,

$$\begin{aligned}
q_n(p) &= \frac{\lambda}{\bar{\bar{\lambda}}} a_{n+1} e^{-b_{n+1}p} \\
\frac{\partial q_n(p)}{\partial p} &= -\frac{\lambda}{\bar{\bar{\lambda}}} a_{n+1} b_{n+1} e^{-b_{n+1}p}.
\end{aligned}$$

With (4.10) we obtain

$$\begin{aligned}
\frac{\partial g_n(p)}{\partial p} &= \frac{\partial q_n(p)}{\partial p}\left[p + V(n+1) - V(n)\right] + q_n(p) \\
&= -\frac{\lambda}{\bar{\bar{\lambda}}} a_{n+1} b_{n+1} e^{-b_{n+1}p}\left[p + V(n+1) - V(n)\right] + \frac{\lambda}{\bar{\bar{\lambda}}} a_{n+1} e^{-b_{n+1}p} \\
&= \frac{\lambda}{\bar{\bar{\lambda}}} a_{n+1} e^{-b_{n+1}p}\{1 - b_{n+1}\left[p + V(n+1) - V(n)\right]\}.
\end{aligned}$$

If there exists an admission fee p^* satisfying $\frac{\partial g_n(p^*)}{\partial p} = 0$, then

$$\begin{aligned}
p_n^* &= \frac{1}{b_{n+1}} + V(n) - V(n+1) \\
&= R\phi^{n+1} + V(n) - V(n+1).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\frac{\partial g_n(p)}{\partial p} &\geq 0, \quad p \leq p^* \\
\frac{\partial g_n(p)}{\partial p} &\leq 0, \quad p \geq p^*.
\end{aligned}$$

□

Theorem 4.4.2. *Suppose X and Y are independent random variables. And $X \sim \exp(\frac{1}{R})$ and $Y \sim U[C-\epsilon, C+\epsilon]$, where $\epsilon \leq C$. Let R, C and ϵ be positive constants. Then the optimal pricing policy at queue length n is*

$$p_n^* = R\phi^{n+1} + V(n) - V(n+1).$$

Furthermore, the Bellman equations are

$$\begin{aligned}
V(0) &= \frac{\mu}{\gamma+v}V(0) + \frac{\bar{\lambda}}{\gamma+v}\left[V(0) + \frac{\lambda}{\bar{\bar{\lambda}}}R\phi\bar{F}_{\Theta_1}\left(R\phi + V(0) - V(1)\right)\right] \\
V(n) &= \frac{\mu}{\gamma+v}V(n-1) + \frac{\bar{\lambda}}{\gamma+v}\Bigg[V(n) \\
&\quad + \frac{\lambda}{\bar{\bar{\lambda}}}R\phi^{n+1}\bar{F}_{\Theta_{n+1}}\left(R\phi^{n+1} + V(n) - V(n+1)\right)\Bigg] \\
&\quad n = 1, 2, \ldots.
\end{aligned}$$

Proof. The derivation is as follows:

$$\begin{aligned}
q_n(p_n^*) &= \frac{\lambda}{\bar{\bar{\lambda}}}a_{n+1}e^{b_{n+1}(V(n+1)-V(n))-1} \\
g_n(p_n^*) &= [1 - q_n(p_n^*)] + q_n(p_n^*)\left[p_n^* + V(n+1)\right] \\
&= V(n) + \left[p_n^* + V(n+1) - V(n)\right] q_n(p_n^*) \\
&= V(n) + R\phi^{n+1}q_n(p_n^*) \\
&= V(n) + \frac{\lambda}{\bar{\bar{\lambda}}}R\phi^{n+1}\bar{F}_{\Theta_n}(R\phi^{n+1} + V(n) - V(n+1))
\end{aligned}$$

$$\begin{aligned}
V(0) &= \frac{\mu}{\gamma+v}V(0) + \frac{\bar{\lambda}}{\gamma+v}\left[V(0) + \frac{\lambda}{\bar{\bar{\lambda}}}R\phi\bar{F}_{\Theta_1}\left(R\phi + V(0) - V(1)\right)\right] \\
V(n) &= \frac{\mu}{\gamma+v}V(n-1) + \frac{\bar{\lambda}}{\gamma+v}\Bigg[V(n) \\
&\quad + \frac{\lambda}{\bar{\bar{\lambda}}}R\phi^{n+1}\bar{F}_{\Theta_{n+1}}\left(R\phi^{n+1} + V(n) - V(n+1)\right)\Bigg] \\
&\quad n = 1, 2, \ldots.
\end{aligned}$$

□

Next we give four computational examples using the algorithm in Section 4.4. We study the sensitivity of the optimal revenue rate and the optimal pricing policy to various parameters. We consider four cases: various service value expectation in case 1; various waiting cost rate expectation in case 2; various uniform cost parameter in case 3; and various discount factor in case 4. The parameters are shown in Table 4.1.

Expo service Uniform cost	$\underline{\lambda}$	$\bar{\lambda}$	μ	γ	R	C	ϵ
Case 1	0.5	1	2	0.9	(10,1010,10)	4	0.5
Case 2	1	1	2	0.9	50	(1,1011,10)	0.5
Case 3	1	1	2	0.9	100	1000	(1,1000,5)
Case 4	1	1	2	(0.5,1,0.05)	15	20	0.5

Table 4.1: Exponential Service Value Uniform Waiting Cost Rate

Figures 4.1, 4.2, 4.3 and 4.4 correspond to case 1. Figures 4.5, 4.6, 4.7 and 4.8 correspond to case 2. Figures 4.9, 4.10, 4.11 and 4.12 correspond to case 3. Figures 4.13, 4.14, 4.15 and 4.16 correspond to case 4.

The service values are sampled from the given distributions. We observe that the optimal revenue rate function $V^*(n)$ appears to be non-increasing in n, based on figures 4.1, 4.5, 4.9 and 4.13.

Similarly, the optimal pricing policy appears to be non-increasing in the queue length n, based on figures 4.2, 4.6, 4.10 and 4.14.

When the waiting cost rate is fixed, p_n^* and $V^*(n)$ are non-decreasing in the service value expectation R (figures 4.1, 4.2). When R is fixed, p_n^* and $V^*(n)$ are non-increasing in the waiting cost rate expectation C (figures 4.5, 4.6), and non-decreasing in the uniform distribution parameter ϵ (figures 4.9, 4.10). When both R and C are fixed, p_n^* and $V^*(n)$ are non-increasing in the discount factor γ (figures 4.13, 4.14).

We consider the following two performance measures of the algorithm: the number of algorithm iterations and the CPU running time. We observe that these two are both non-decreasing in R (figures 4.3, 4.4) and ϵ (figures 4.11, 4.12), and non-increasing in C (figures 4.7, 4.8) and γ (figures 4.15, 4.16).

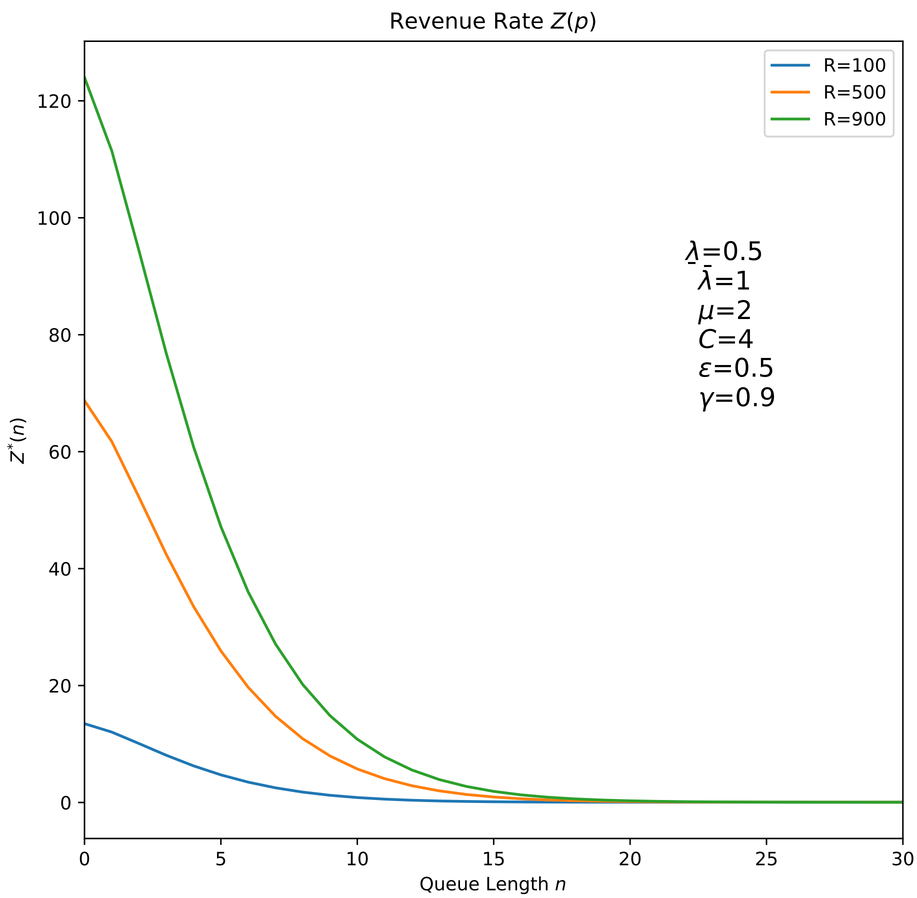

Figure 4.1: Expo service Uniform cost, Revenue Rate Optimality, case 1, Various Expected Service Value, Revenue Rate

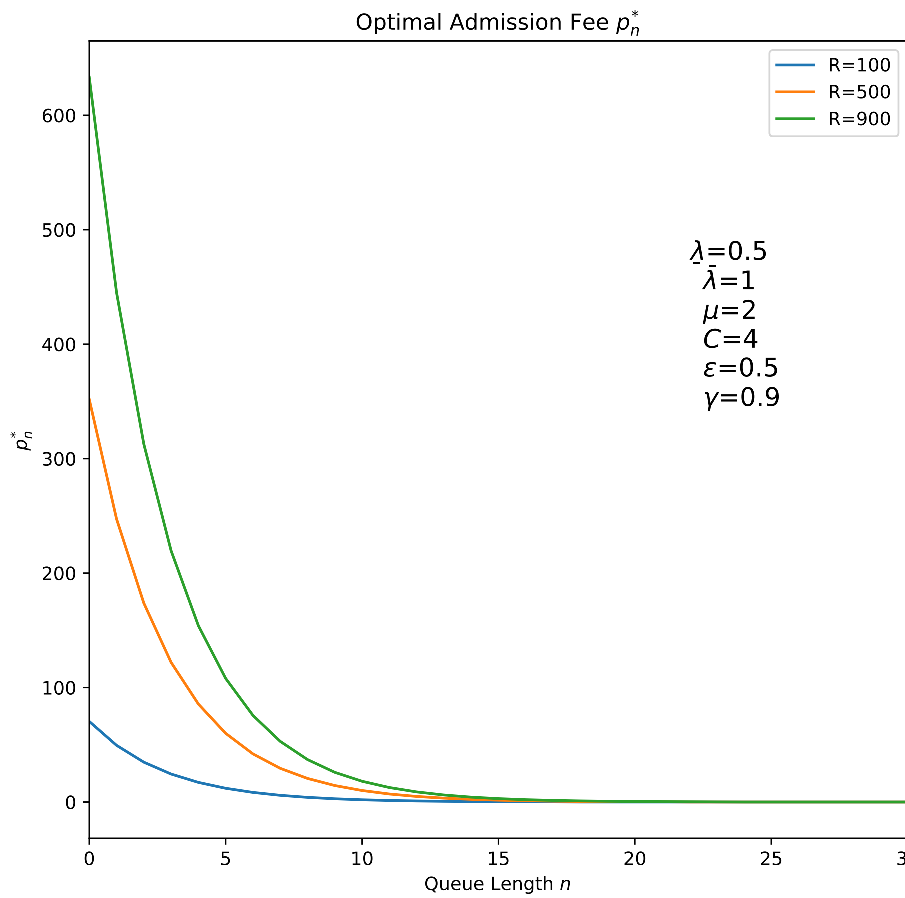

Figure 4.2: Expo service Uniform cost, Revenue Rate Optimality, case 1, Various Expected Service Value, Optimal Fee

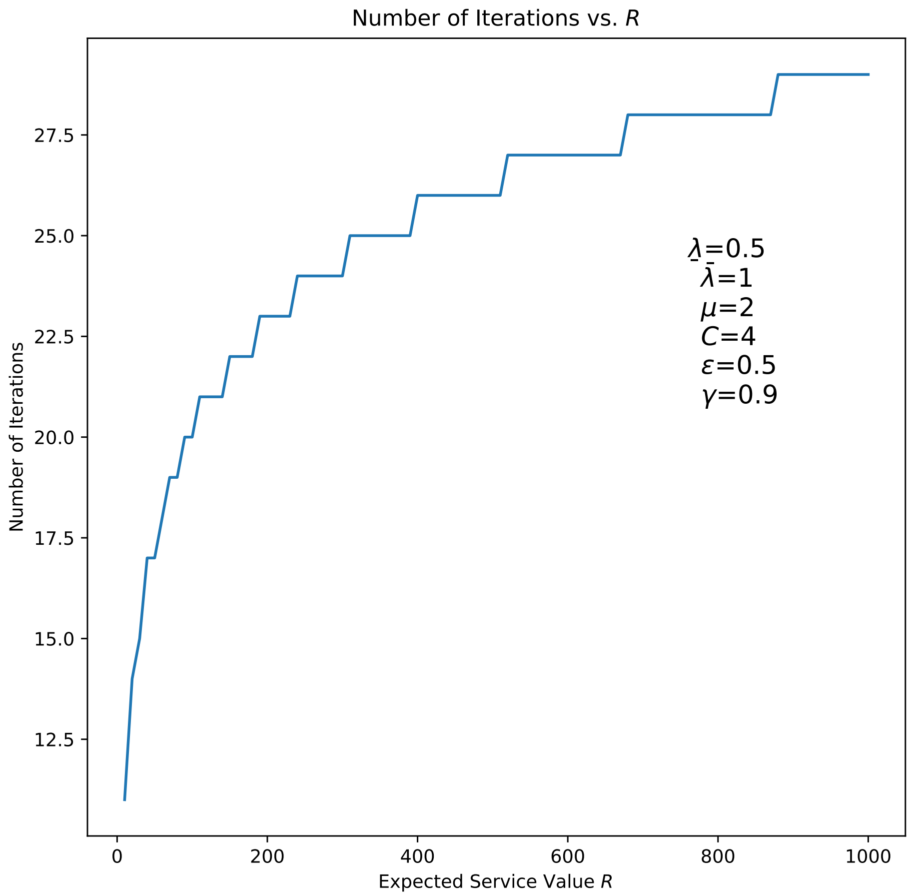

Figure 4.3: Expo service Uniform cost, Revenue Rate Optimality, case 1, Various Expected Service Value, Algorithm Iterations

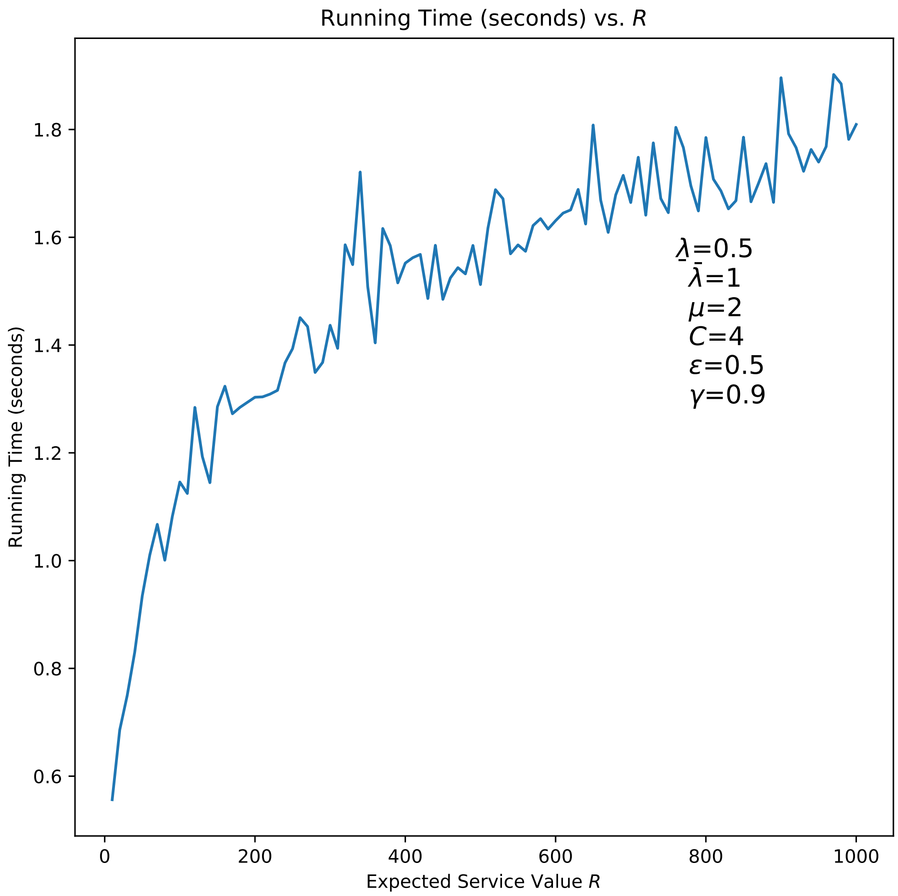

Figure 4.4: Expo service Uniform cost, Revenue Rate Optimality, case 1, Various Expected Service Value, Execution Time

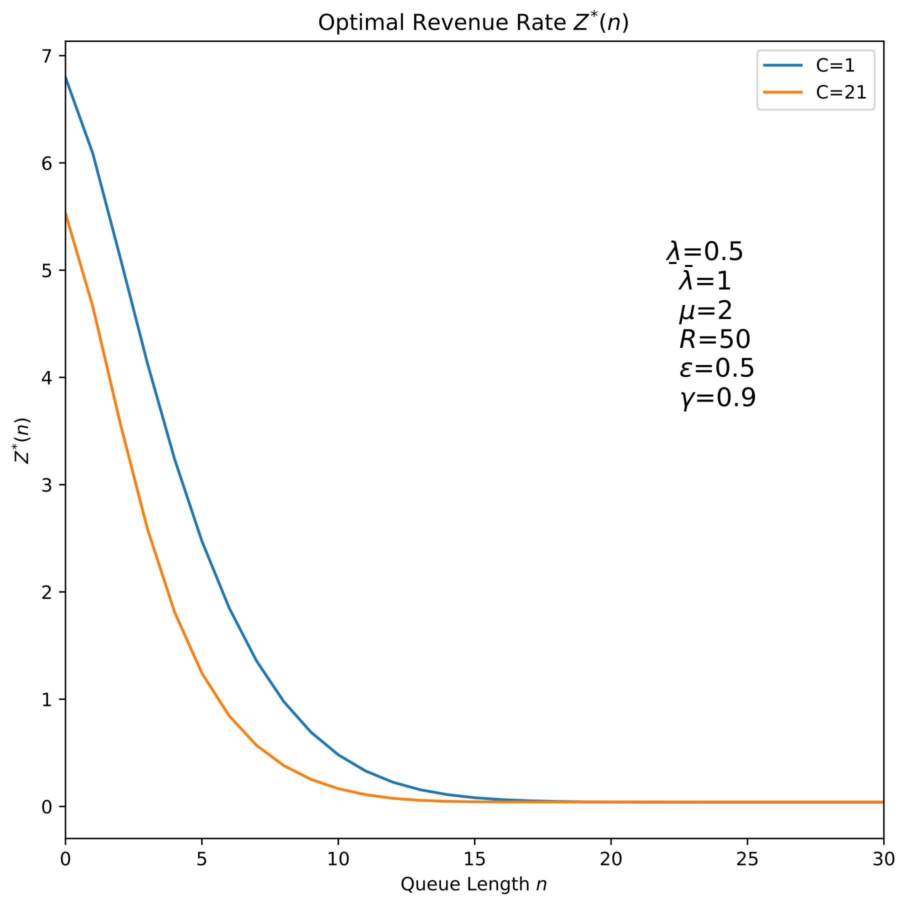

Figure 4.5: Expo service Uniform cost, Revenue Rate Optimality, case 2, Various Expected Waiting Cost Rate, Revenue Rate

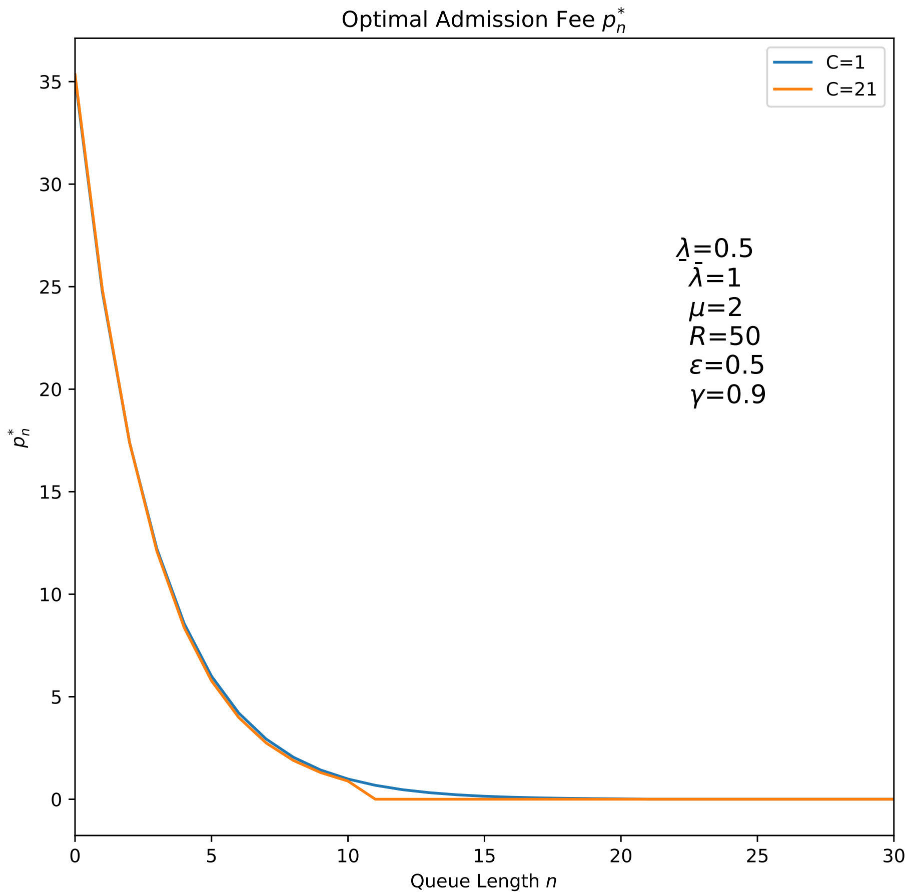

Figure 4.6: Expo service Uniform cost, Revenue Rate Optimality, case 2, Various Expected Waiting Cost Rate, Optimal Fee

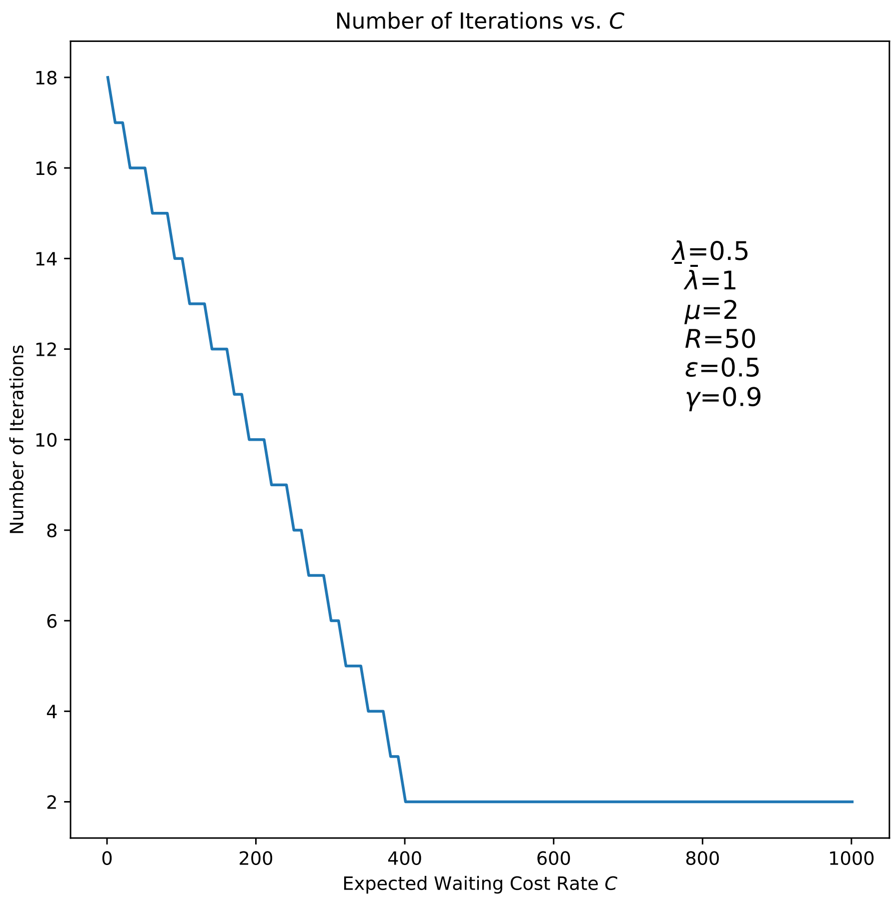

Figure 4.7: Expo service Uniform cost, Revenue Rate Optimality, case 2, Various Expected Waiting Cost Rate, Algorithm Iterations

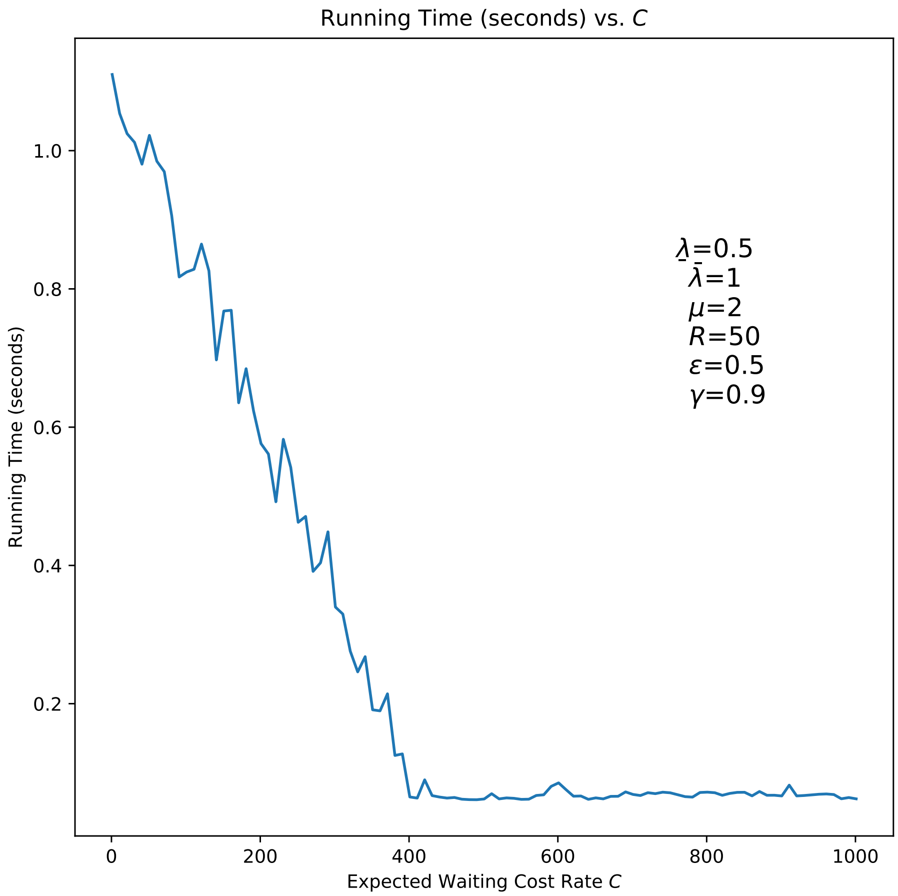

Figure 4.8: Expo service Uniform cost, Revenue Rate Optimality, case 2, Various Expected Waiting Cost Rate, Execution Time

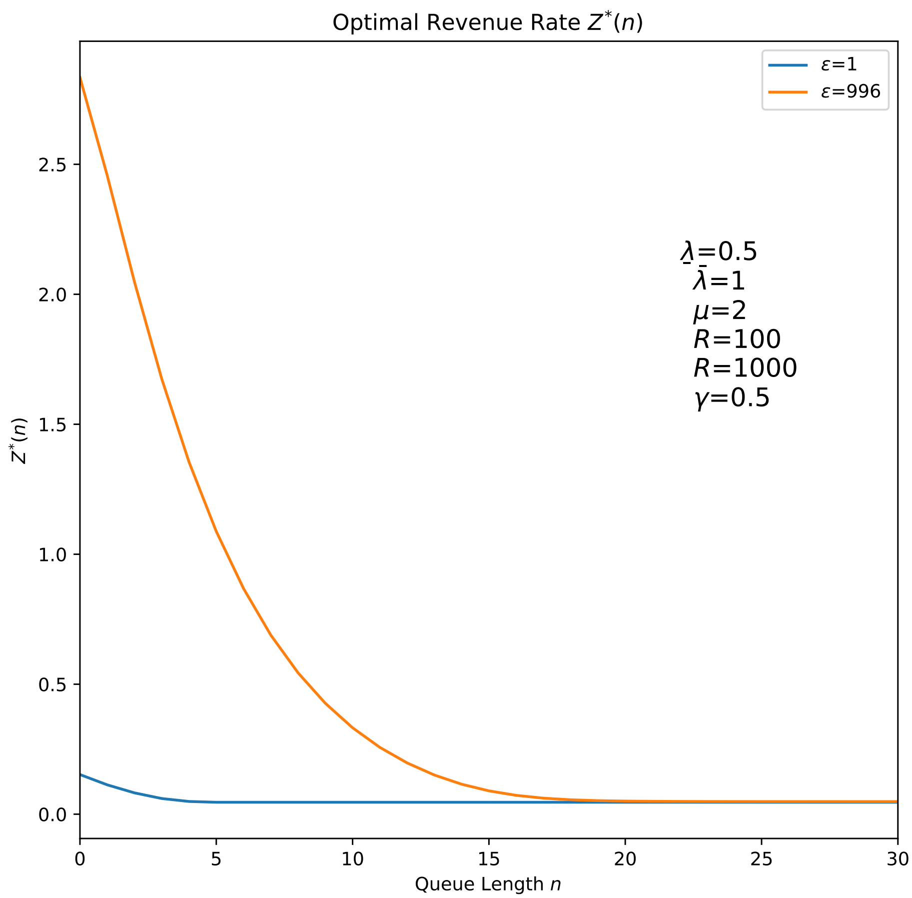

Figure 4.9: Expo service Uniform cost, Revenue Rate Optimality, case 3, Various Uniform Parameter, Revenue Rate

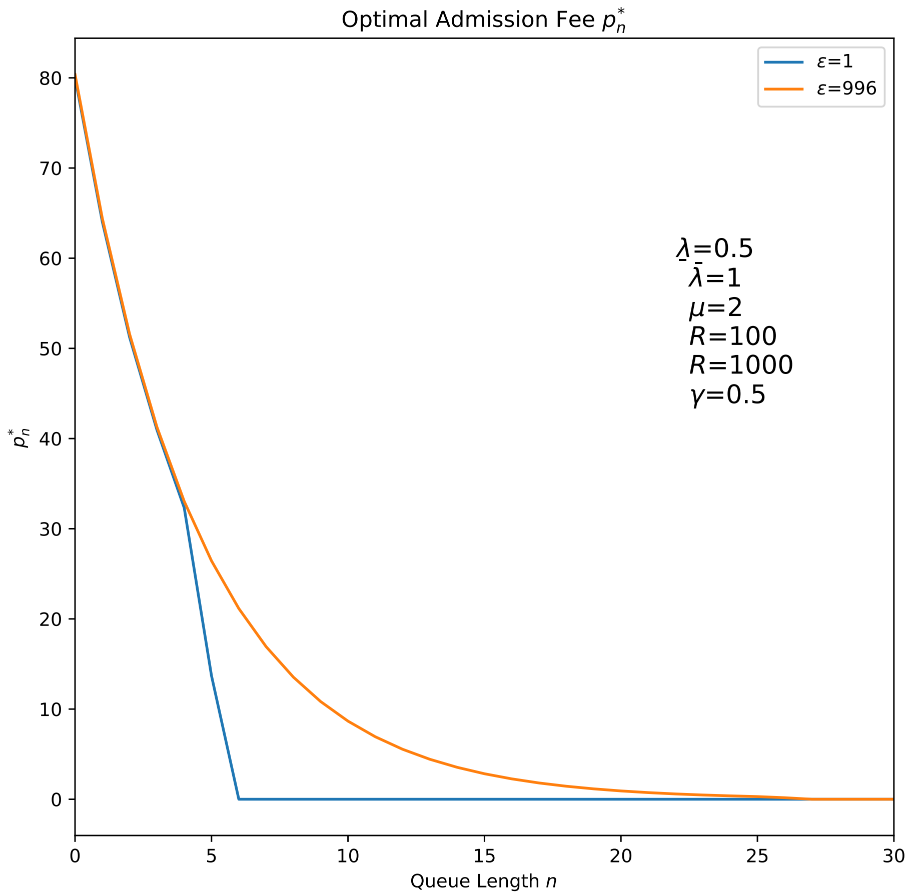

Figure 4.10: Expo service Uniform cost, Revenue Rate Optimality, case 3, Various Uniform Parameter, Optimal Fee

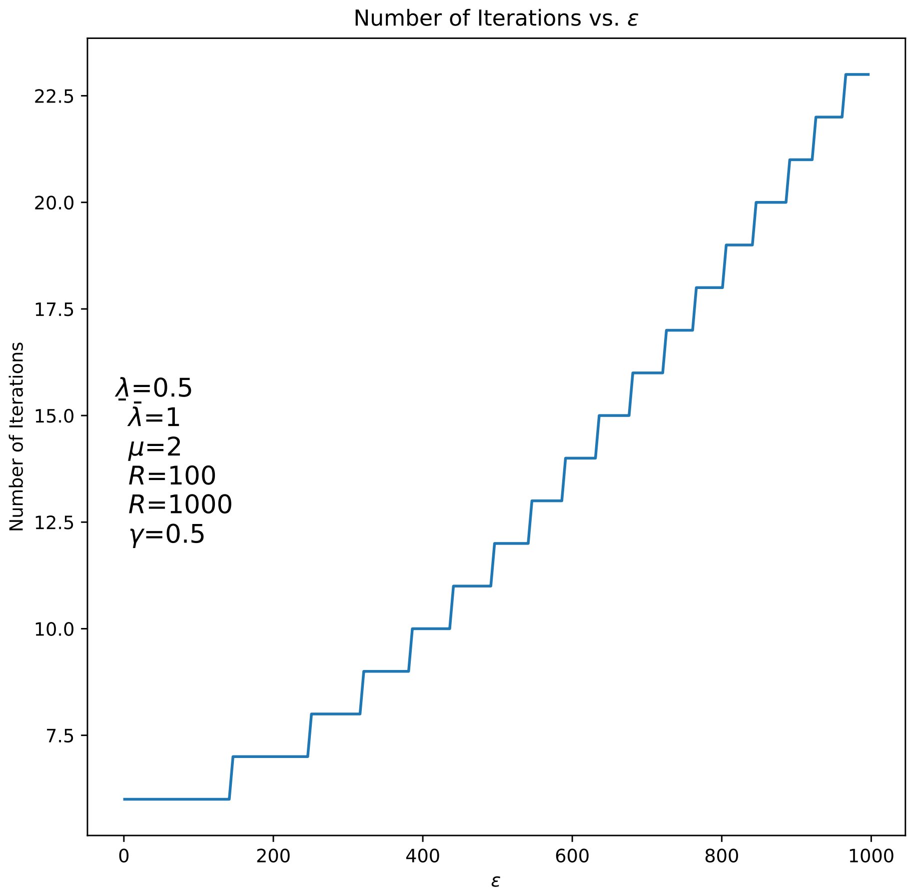

Figure 4.11: Expo service Uniform cost, Revenue Rate Optimality, case 3, Various Uniform Parameter, Algorithm Iterations

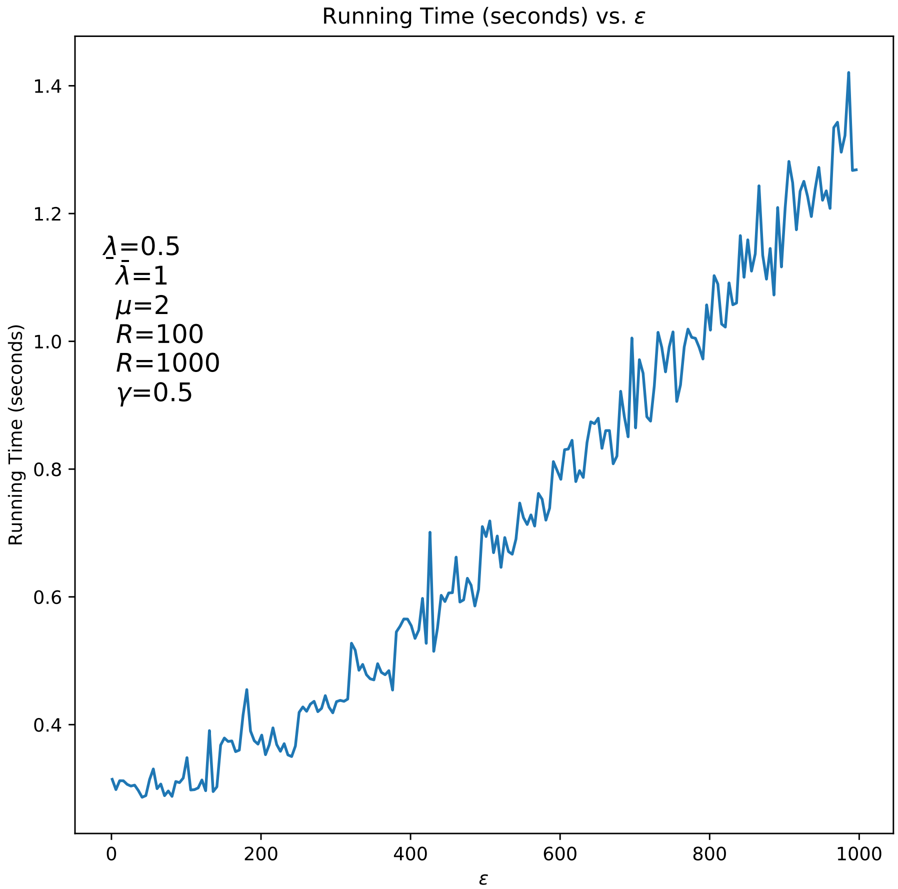

Figure 4.12: Expo service Uniform cost, Revenue Rate Optimality, case 3, Various Uniform Parameter, Execution Time

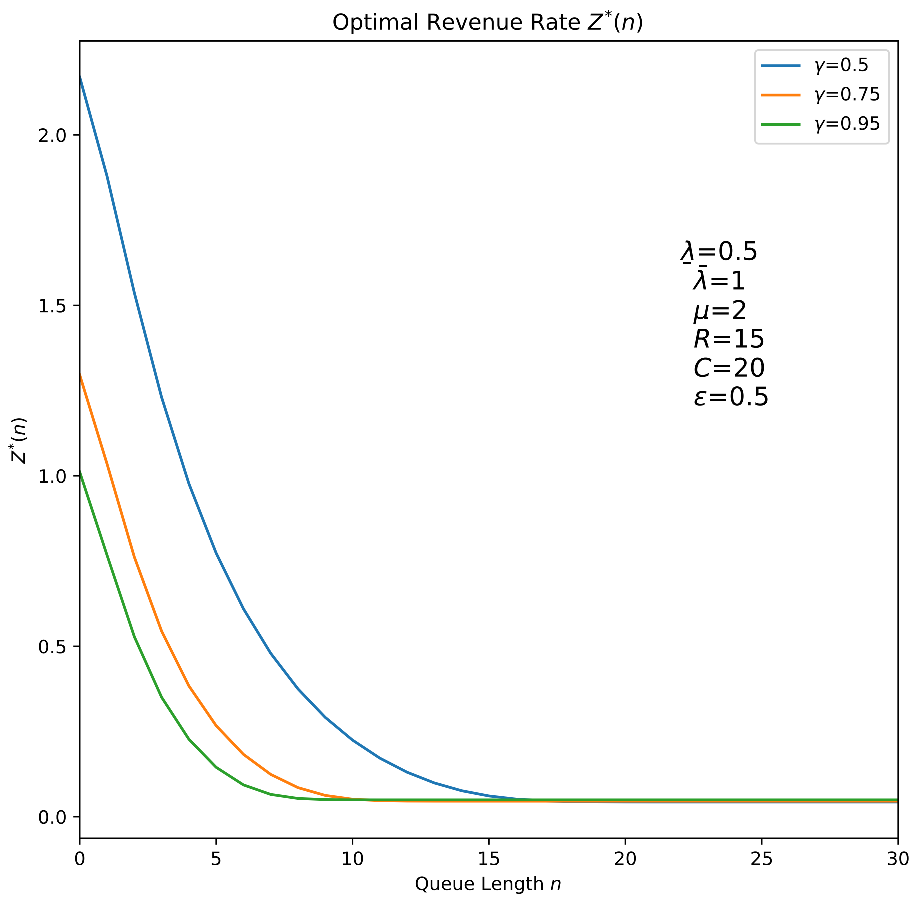

Figure 4.13: Expo service Uniform cost, Revenue Rate Optimality, case 4, Various Discount Factor, Revenue Rate

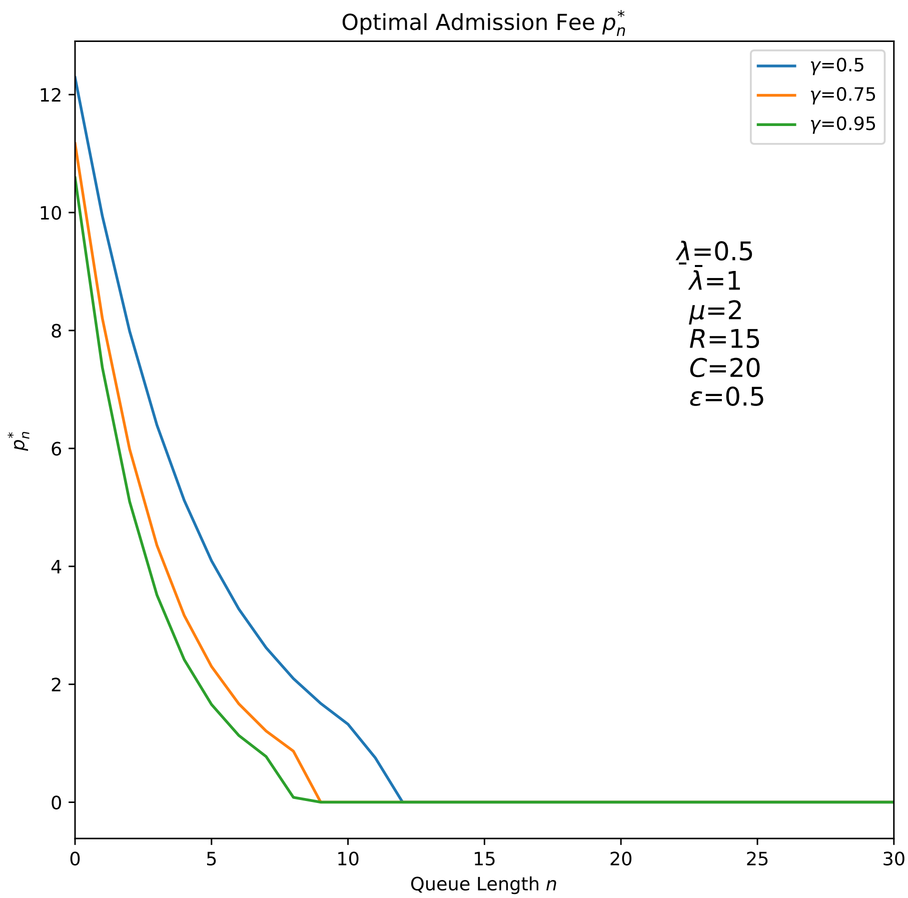

Figure 4.14: Expo service Uniform cost, Revenue Rate Optimality, case 4, Various Discount Factor, Optimal Fee

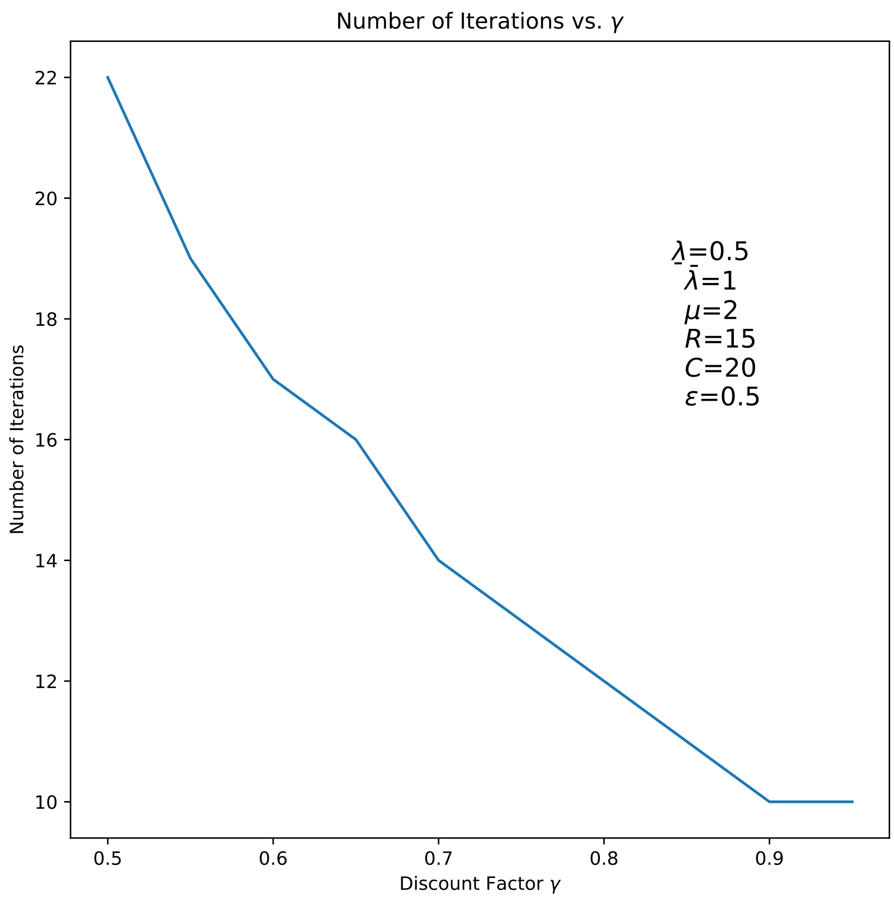

Figure 4.15: Expo service Uniform cost, Revenue Rate Optimality, case 4, Various Discount Factor, Algorithm Iterations

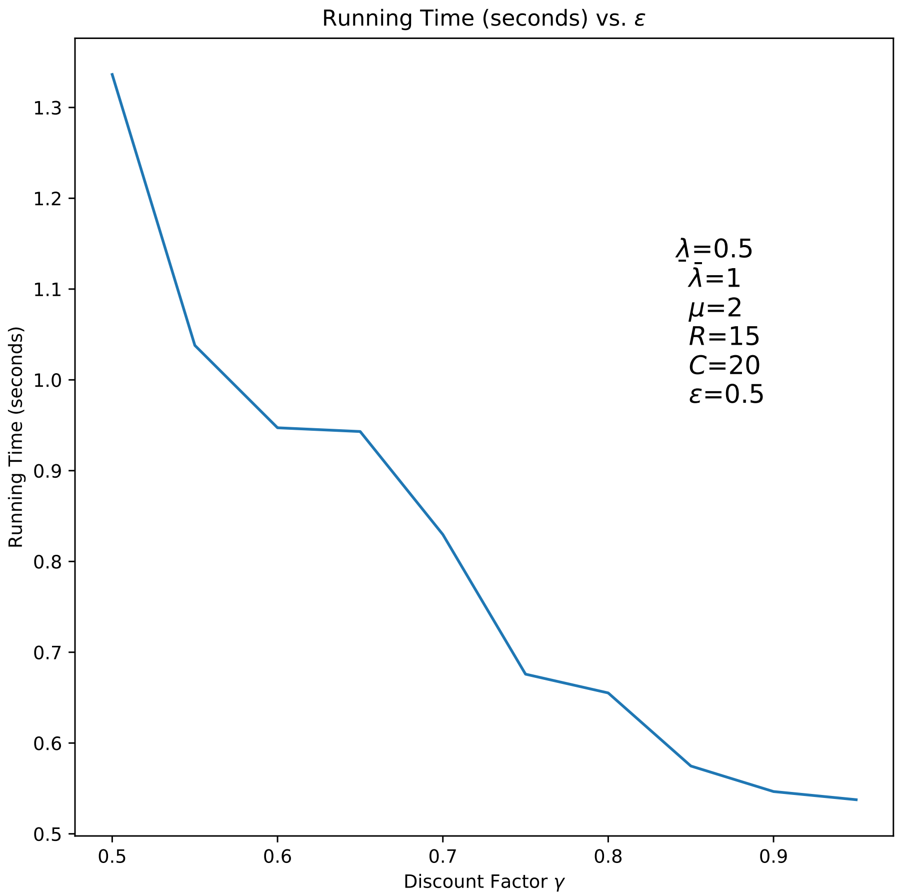

Figure 4.16: Expo service Uniform cost, Revenue Rate Optimality, case 4, Various Discount Factor, Execution Time

4.4.2 Pareto Service Uniform Cost

Assume the service value X and the waiting cost rate Y are independent. In this section we discuss a case when X has a Pareto distribution with parameter α, and $Y \sim U[C-\epsilon, C+\epsilon]$, where α, C and ϵ are positive constants. Then $\alpha = \frac{R}{R-1}$, given $R > 1$.

We first derive he joint pdf of X and Y:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} \frac{\alpha}{2\epsilon x^{\alpha+1}} & x \geq 1 \\ 0 & x < 1. \end{cases}$$

The joining fraction of the customer population is

$$\begin{aligned} \bar{F}_{\Theta_n}(p) &= \Pr_{X,Y}\left(\phi^n X - \frac{1-\phi^n}{\gamma}Y \geq p\right) \\ &= \iint_{\substack{\phi^n x - \frac{1-\phi^n}{\gamma} y \geq p \\ C-\epsilon \leq y \leq C+\epsilon \\ x > \geq 1}} f_{X,Y}(x,y)dxdy. \end{aligned}$$

If

$$\phi^n > \frac{C+\epsilon+\gamma p}{C+\epsilon+\gamma},$$

then

$$\bar{F}_{\Theta_n}(p) = \frac{1}{2\epsilon}\int_{C-\epsilon}^{C+\epsilon}\int_1^\infty \frac{\alpha}{x^{\alpha+1}}dxdy = 1.$$

If

$$\phi^n < \frac{C-\epsilon+\gamma p}{C-\epsilon+\gamma},$$

then

$$\bar{F}_{\Theta_n}(p) = \frac{1}{2\epsilon}\int_{C-\epsilon}^{C+\epsilon}\int_{\frac{1-\phi^n}{\gamma\phi^n}y+\frac{p}{\phi^n}}^{\infty}\frac{\alpha}{x^{\alpha+1}}dxdy$$
$$= \frac{(\gamma\phi^n)^\alpha}{2\epsilon(1-\alpha)\,(1-\phi^n)}\Bigg[\left[(1-\phi^n)(C+\epsilon)+\gamma p\right]^{1-\alpha} - \left[(1-\phi^n)(C-\epsilon)+\gamma p\right]^{1-\alpha}\Bigg].$$

If

$$\frac{C-\epsilon+\gamma p}{C-\epsilon+\gamma} \le \phi^n \le \frac{C+\epsilon+\gamma p}{C+\epsilon+\gamma},$$

then

$$\bar{F}_{\Theta_n}(p) = \frac{1}{2\epsilon}\int_{C-\epsilon}^{\frac{\gamma(\phi^n-p)}{1-\phi^n}}\int_{1}^{\infty}\frac{\alpha}{x^{\alpha+1}}dxdy$$
$$\frac{1}{2\epsilon}\int_{\frac{\gamma(\phi^n-p)}{1-\phi^n}}^{C+\epsilon}\int_{\frac{1-\phi^n}{\gamma\phi^n}y+\frac{p}{\phi^n}}^{\infty}\frac{\alpha}{x^{\alpha+1}}dxdy$$
$$= \frac{1}{2\epsilon}\left[\frac{\gamma(\phi^n-p)}{1-\phi^n}-(C-\epsilon)\right] + \frac{(\gamma\phi^n)^\alpha}{2\epsilon(1-\alpha)\,(1-\phi^n)}\Bigg[\left[(1-\phi^n)(C+\epsilon)+\gamma p\right]^{1-\alpha} - \left[\gamma(\phi^n-p)+\gamma p\right]^{1-\alpha}\Bigg].$$

Next we give four computational examples. Similar to section 4.4.1, We study the sensitivity of the optimal revenue rate and the optimal pricing policy to four parameters: service value expectation, waiting cost rate expectation, uniform cost parameter and discount factor. The parameters are shown in Table 4.2.

Figures 4.17, 4.18, 4.19 and 4.20 correspond to case 1. Figures 4.21, 4.22, 4.23 and 4.24 correspond to case 2. Figures 4.25, 4.26, 4.27 and 4.28 correspond to case 3. Figures 4.29, 4.30, 4.31 and 4.32 correspond to case 4.

Pareto service Uniform cost	$\underline{\lambda}$	$\bar{\lambda}$	μ	γ	R	C	ϵ
Case 1	0.5	1	2	0.9	(2,22,1)	4	0.5
Case 2	0.5	1	2	0.9	3	(1,15,0.5)	0.5
Case 3	0.5	1	2	0.9	2.5	10	(0.5,10,0.5)
Case 4	0.5	1	2	(0.5,1,0.05)	5	10	0.5

Table 4.2: Pareto Service Value Uniform Waiting Cost Rate

The service values are sampled from the given distributions. We observe that the optimal revenue rate function $V^*(n)$ appears to be non-increasing in n, based on figures 4.17, 4.21, 4.25 and 4.29.

However, we observe that the optimal pricing policy p_n^* appears to be non-decreasing in the queue length n, based on figures 4.18, 4.22, 4.26 and 4.30. This observation is different than that in section 4.4.1.

When the waiting cost rate is fixed, p_n^* and $V^*(n)$ are non-decreasing in the service value expectation R (figures 4.17, 4.18). When R is fixed, p_n^* is non-increasing in the waiting cost rate expectation C (figure 4.22), but $V^*(n)$ seems to be non-increasing in C when the queue length is below certain value (figure 4.21).

When R and C are fixed, p_n^* and $V^*(n)$ are non-decreasing in the uniform distribution parameter ϵ (figures 4.25, 4.26). And p_n^* and $V^*(n)$ are non-increasing in the discount factor γ (figures 4.29, 4.30).

In terms of the number of algorithm iterations and the CPU running time. We observe that these two are both non-decreasing in R (figures 4.19, 4.20). The affect of ϵ is not obvious from figures 4.27, 4.28. This is different than the observation about ϵ in section 4.4.1. The iteration numbers and CPU running time are non-increasing in C (figures 4.23, 4.24) and γ (figures 4.31, 4.32).

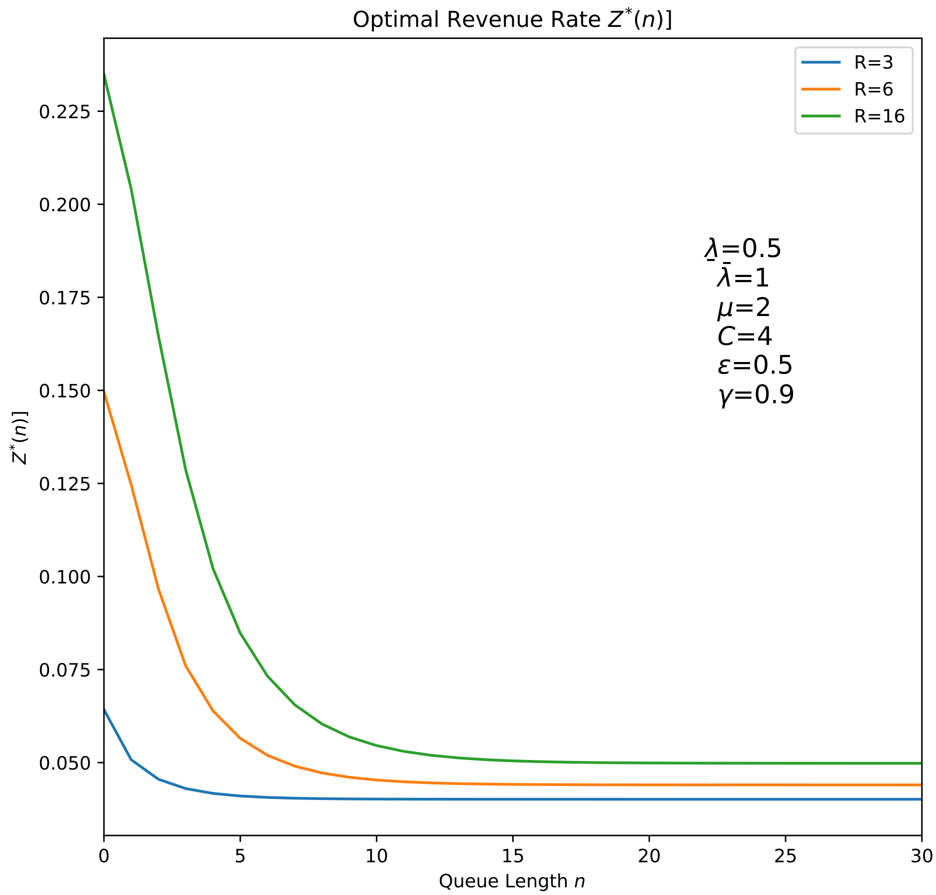

Figure 4.17: Pareto service Uniform cost, Revenue Rate Optimality, case 1, Various Expected Service Value, Revenue Rate

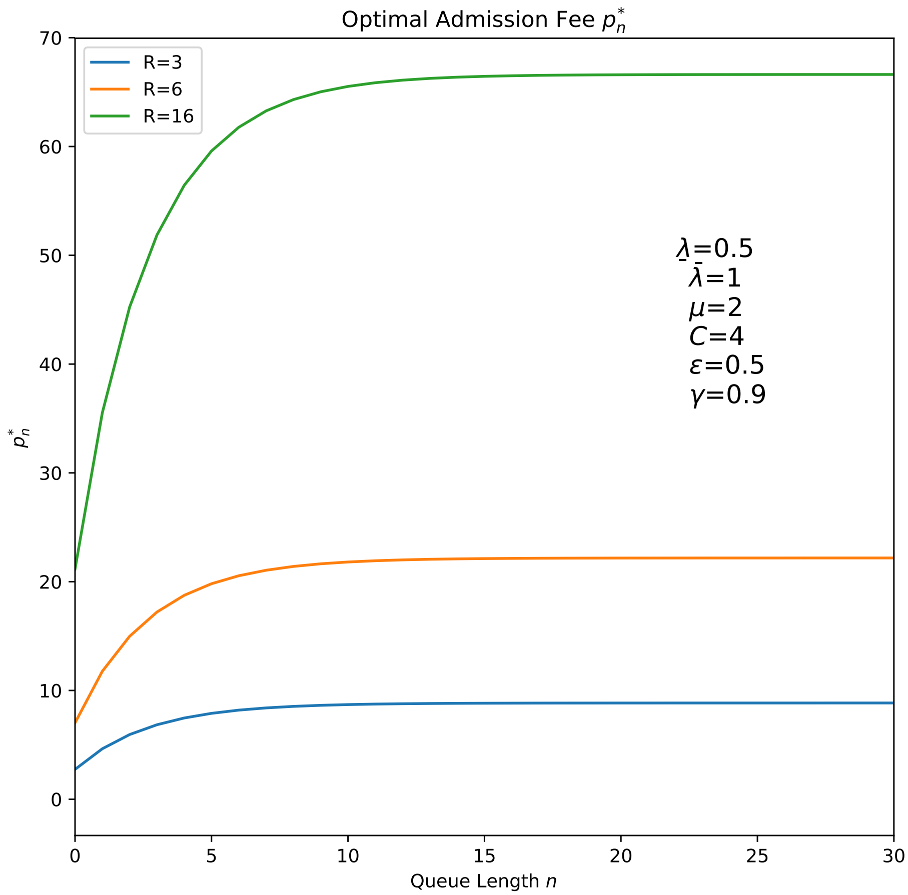

Figure 4.18: Pareto service Uniform cost, Revenue Rate Optimality, case 1, Various Expected Service Value, Optimal Fee

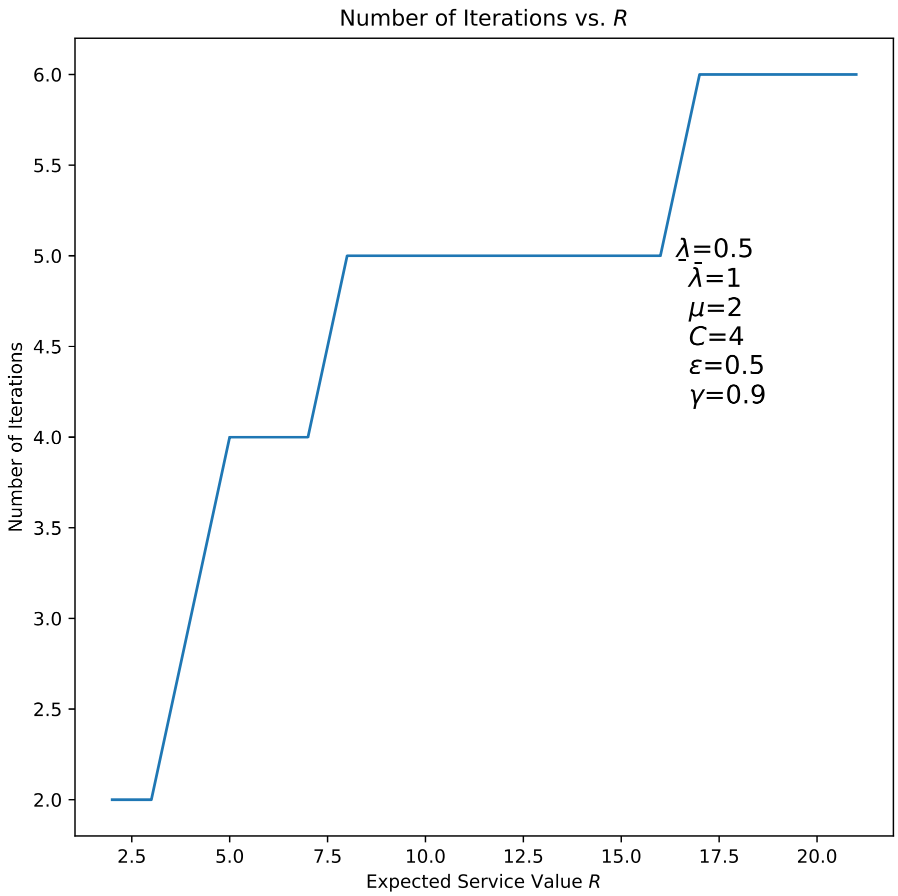

Figure 4.19: Pareto service Uniform cost, Revenue Rate Optimality, case 1, Various Expected Service Value, Algorithm Iterations

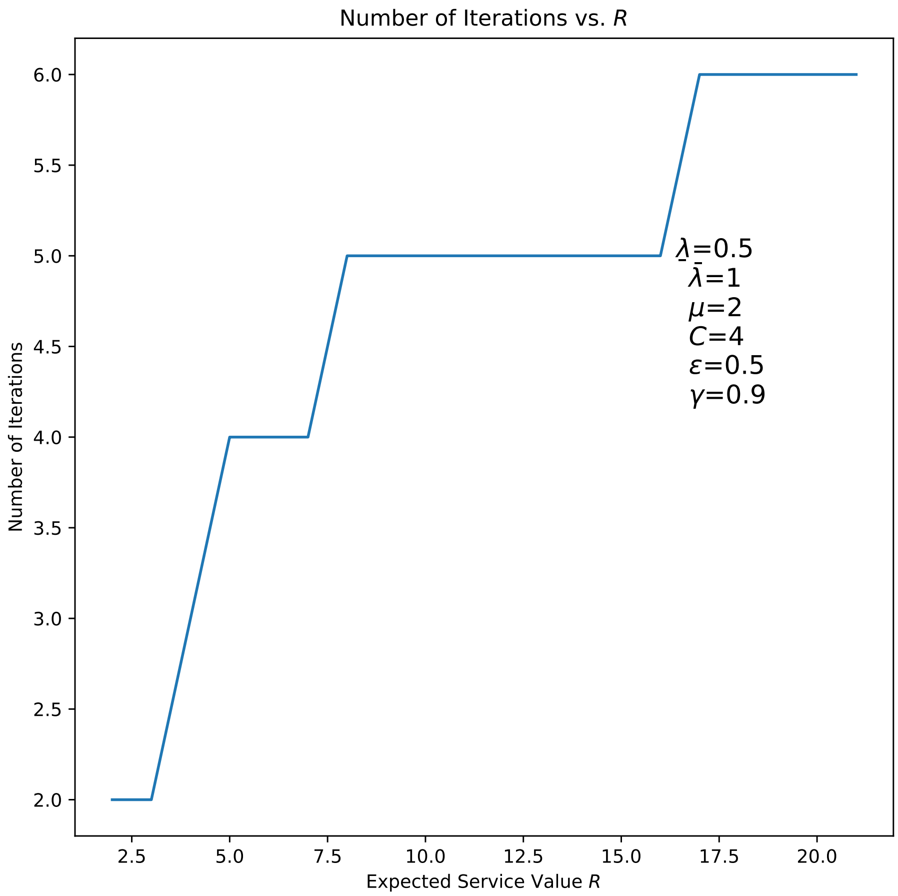

Figure 4.20: Pareto service Uniform cost, Revenue Rate Optimality, case 1, Various Expected Service Value, Execution Time

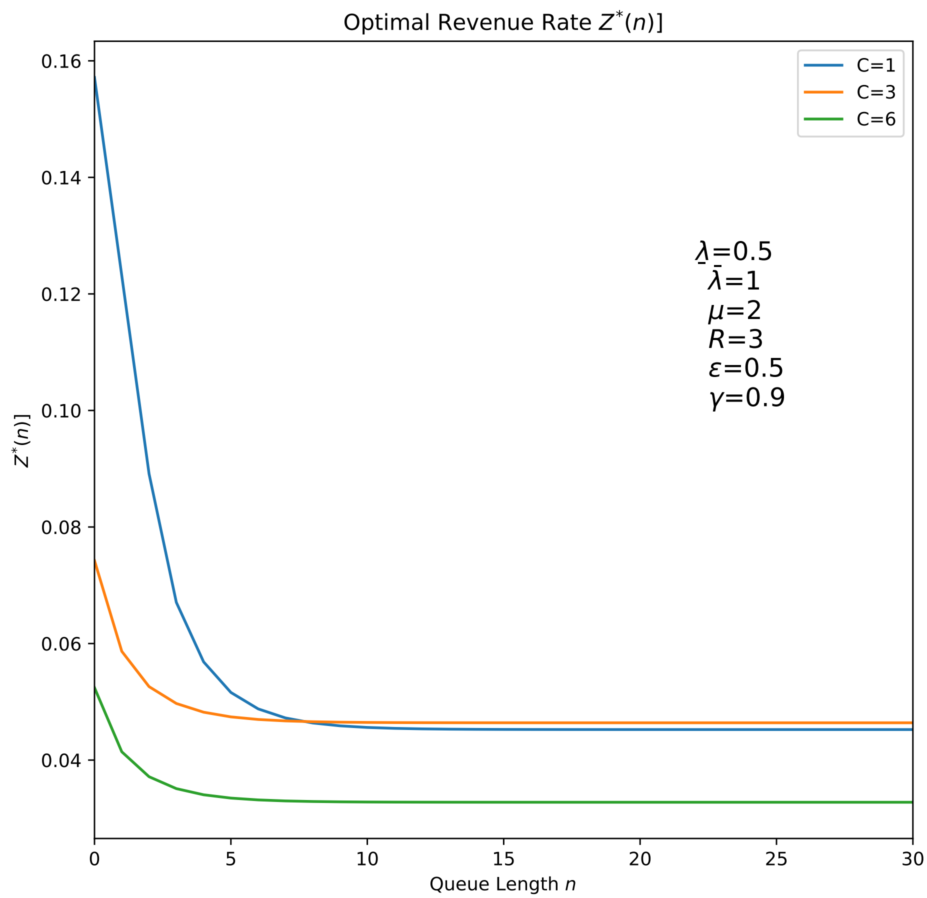

Figure 4.21: Pareto service Uniform cost, Revenue Rate Optimality, case 2, Various Expected Waiting Cost Rate, Revenue Rate

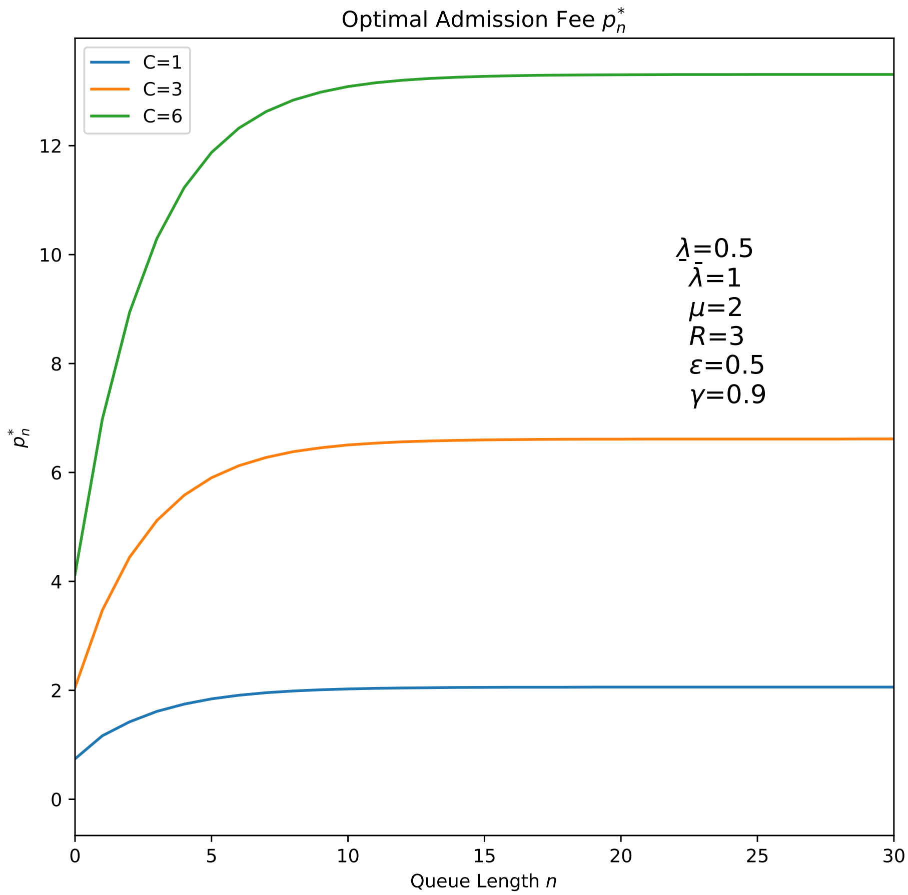

Figure 4.22: Pareto service Uniform cost, Revenue Rate Optimality, case 2, Various Expected Waiting Cost Rate, Optimal Fee

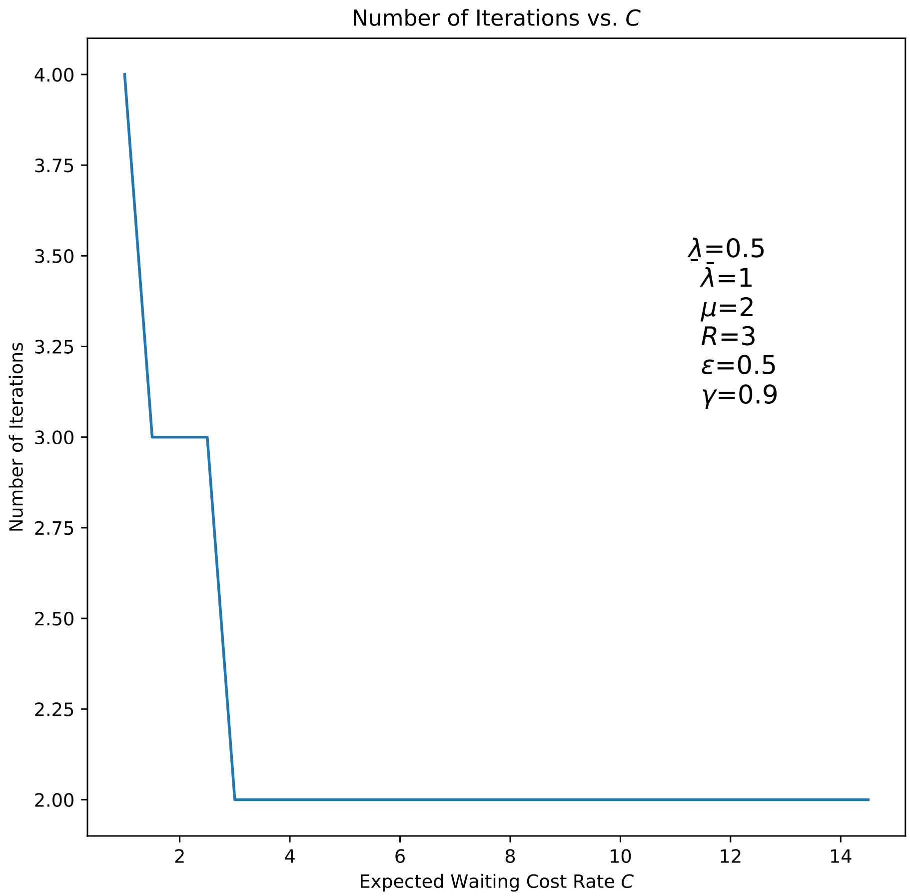

Figure 4.23: Pareto service Uniform cost, Revenue Rate Optimality, case 2, Various Expected Waiting Cost Rate, Algorithm Iterations

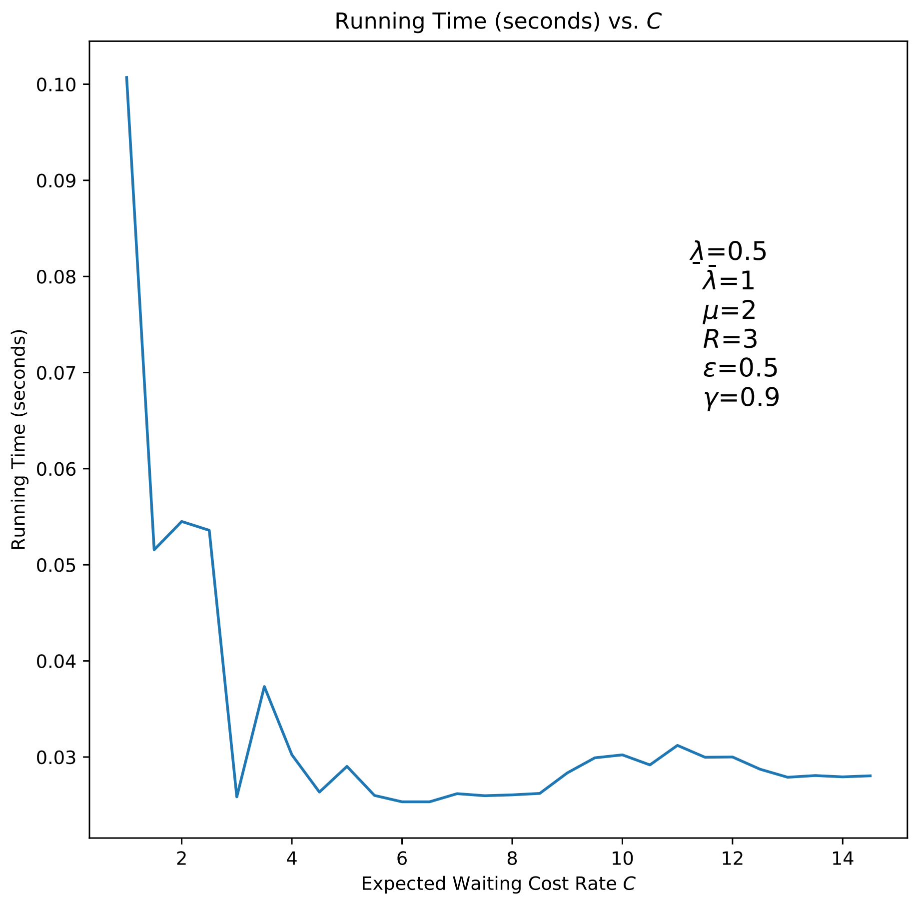

Figure 4.24: Pareto service Uniform cost, Revenue Rate Optimality, case 2, Various Expected Waiting Cost Rate, Execution Time

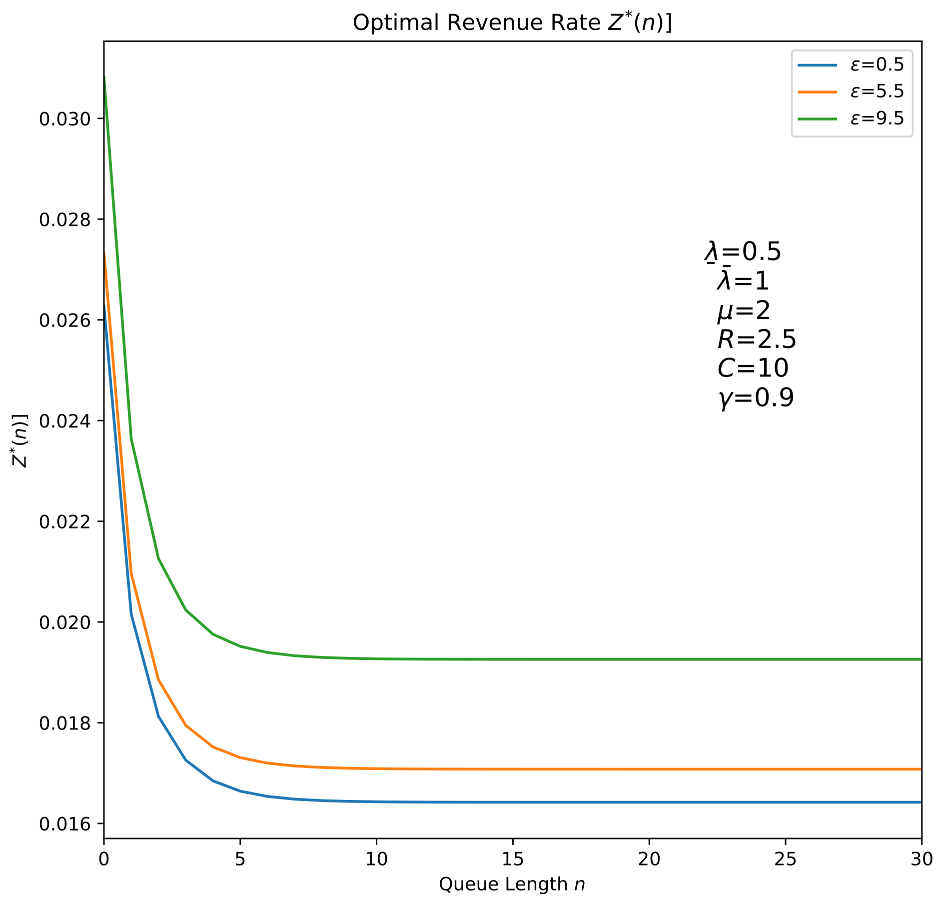

Figure 4.25: Pareto service Uniform cost, Revenue Rate Optimality, case 3, Various Uniform Parameter, Revenue Rate

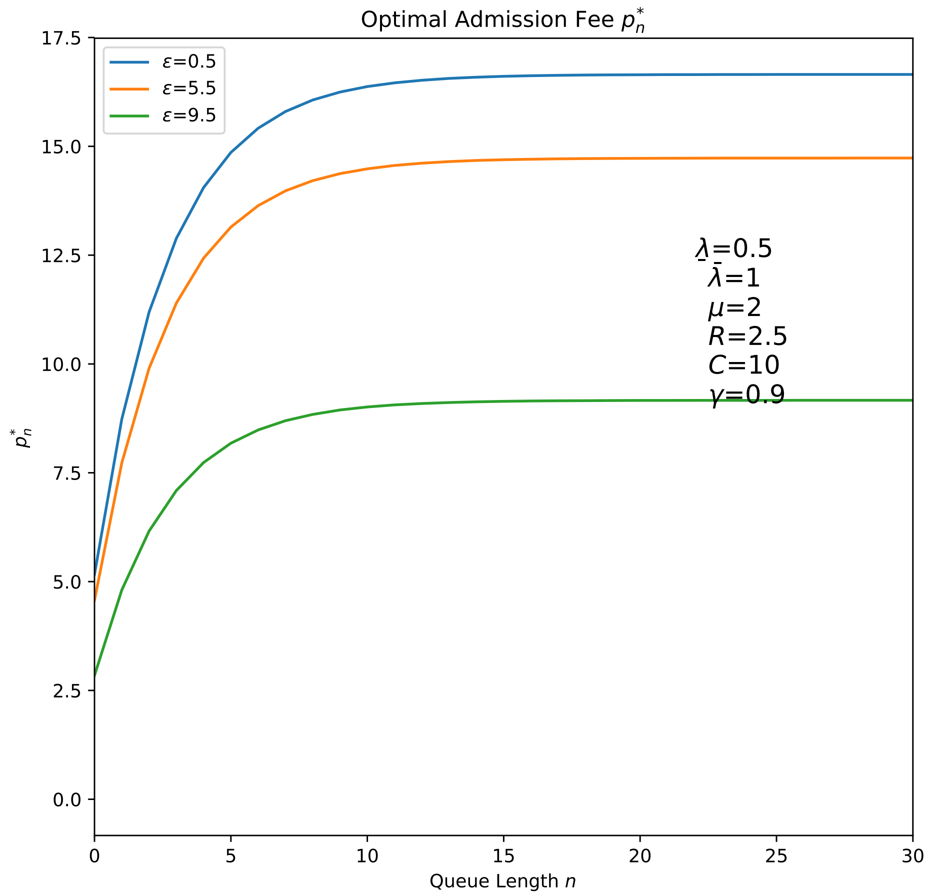

Figure 4.26: Expo service Uniform cost, Revenue Rate Optimality, case 3, Various Uniform Parameter, Optimal Fee

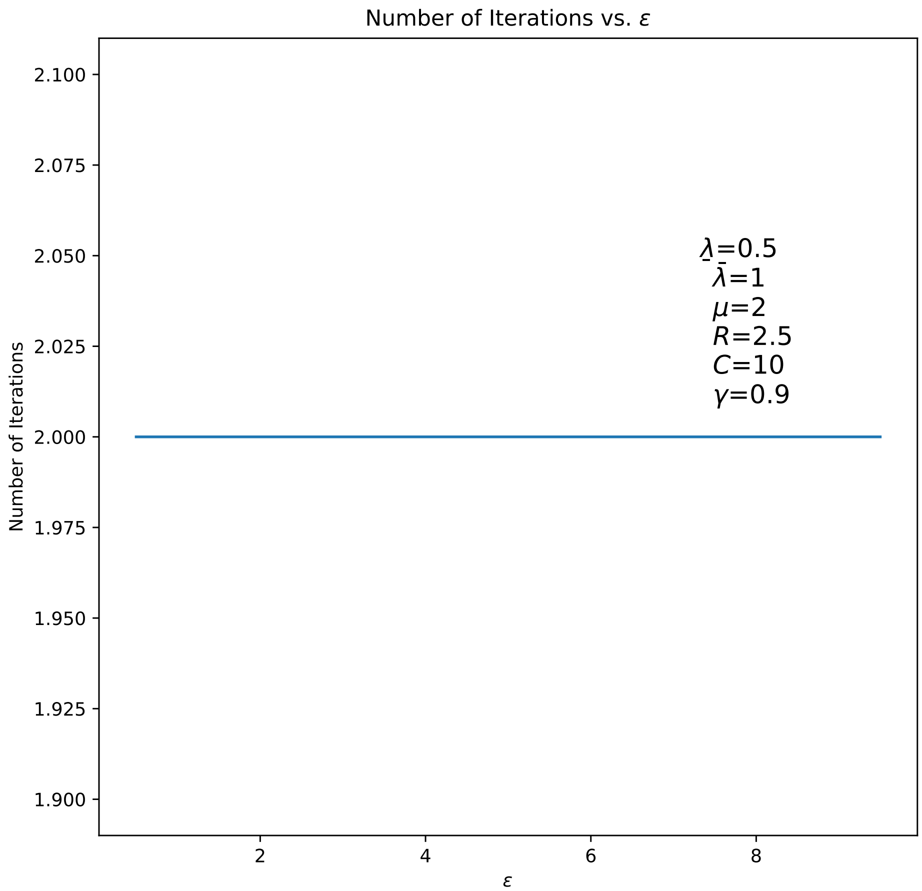

Figure 4.27: Expo service Uniform cost, Revenue Rate Optimality, case 3, Various Uniform Parameter, Algorithm Iterations

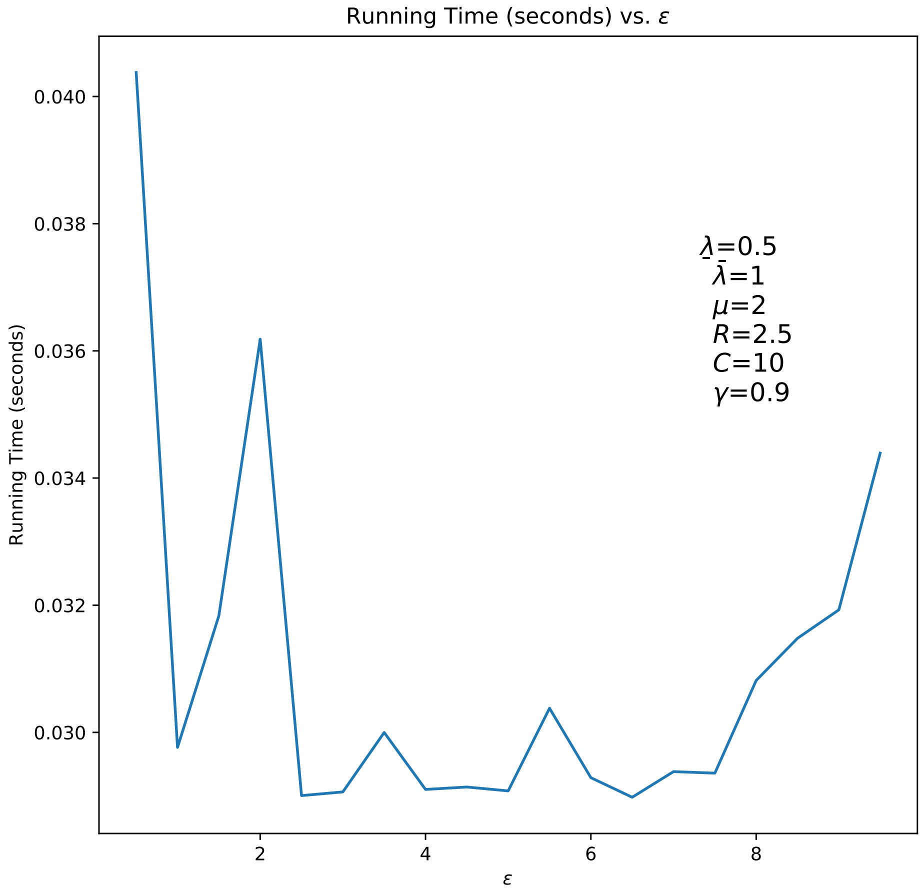

Figure 4.28: Expo service Uniform cost, Revenue Rate Optimality, case 3, Various Uniform Parameter, Execution Time

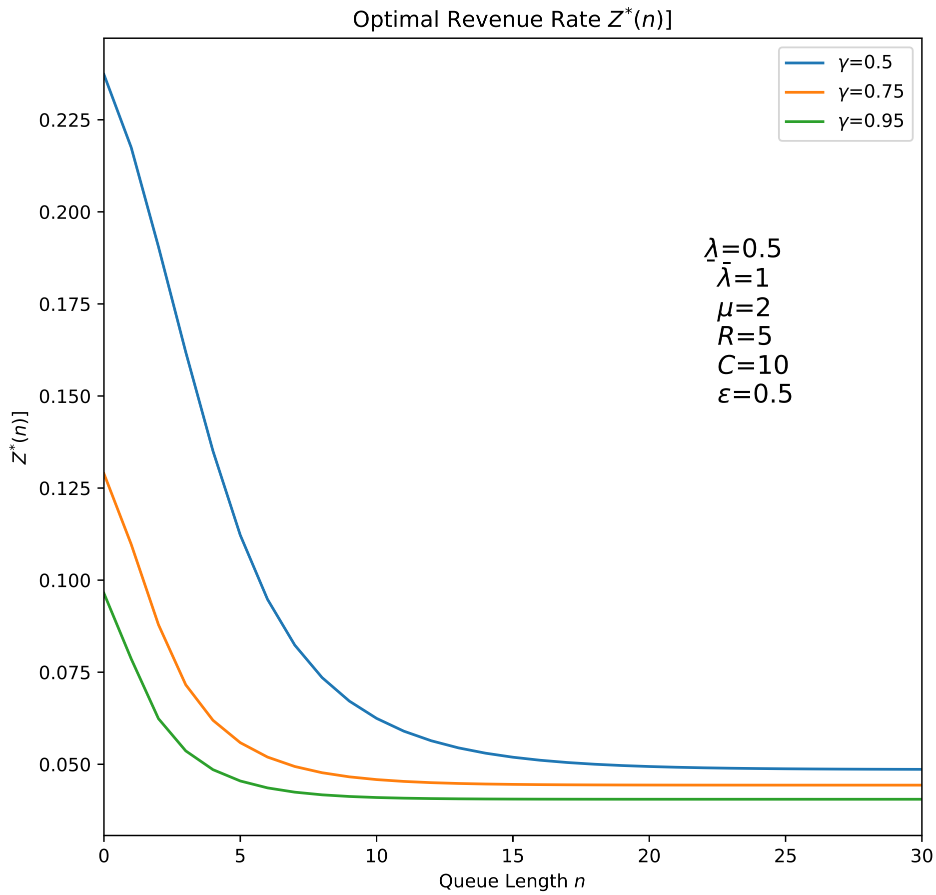

Figure 4.29: Pareto service Uniform cost, Revenue Rate Optimality, case 4, Various Discount Factor, Revenue Rate

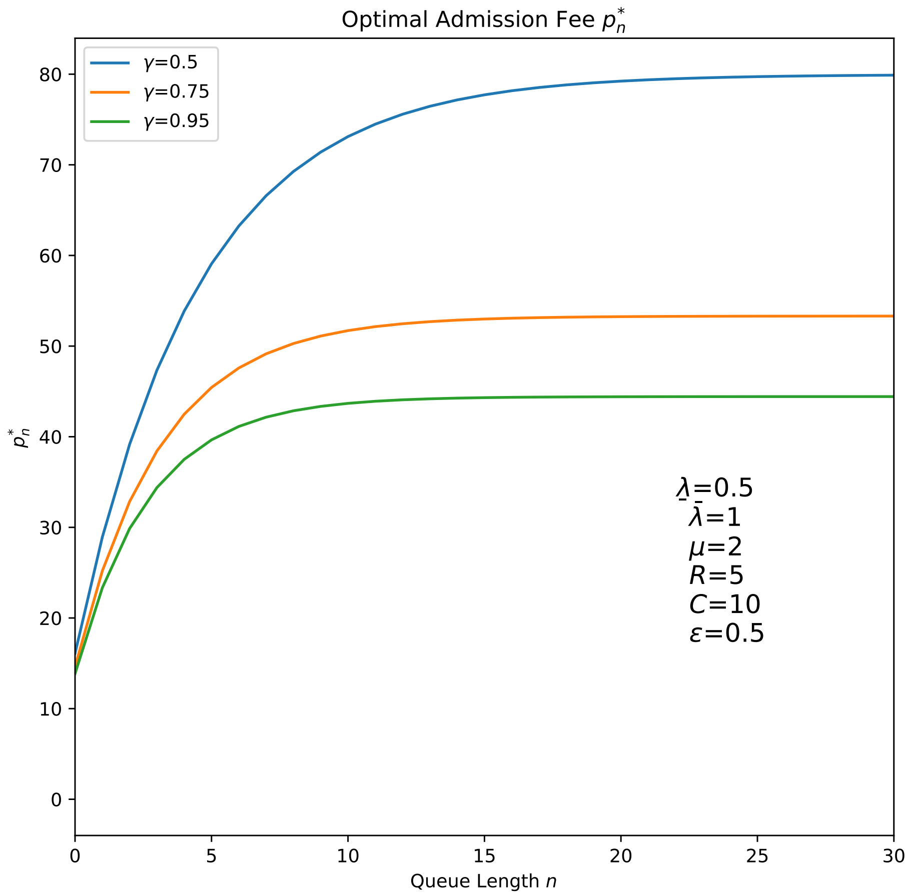

Figure 4.30: Pareto service Uniform cost, Revenue Rate Optimality, case 4, Various Discount Factor, Optimal Fee

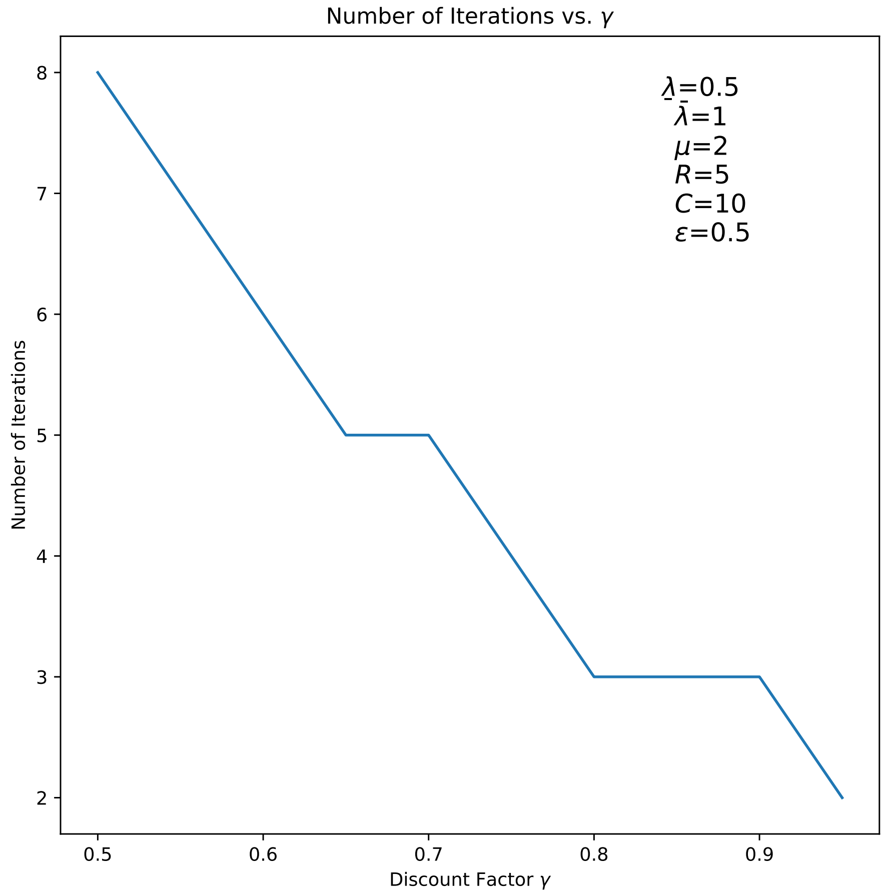

Figure 4.31: Pareto service Uniform cost, Revenue Rate Optimality, case 4, Various Discount Factor, Algorithm Iterations

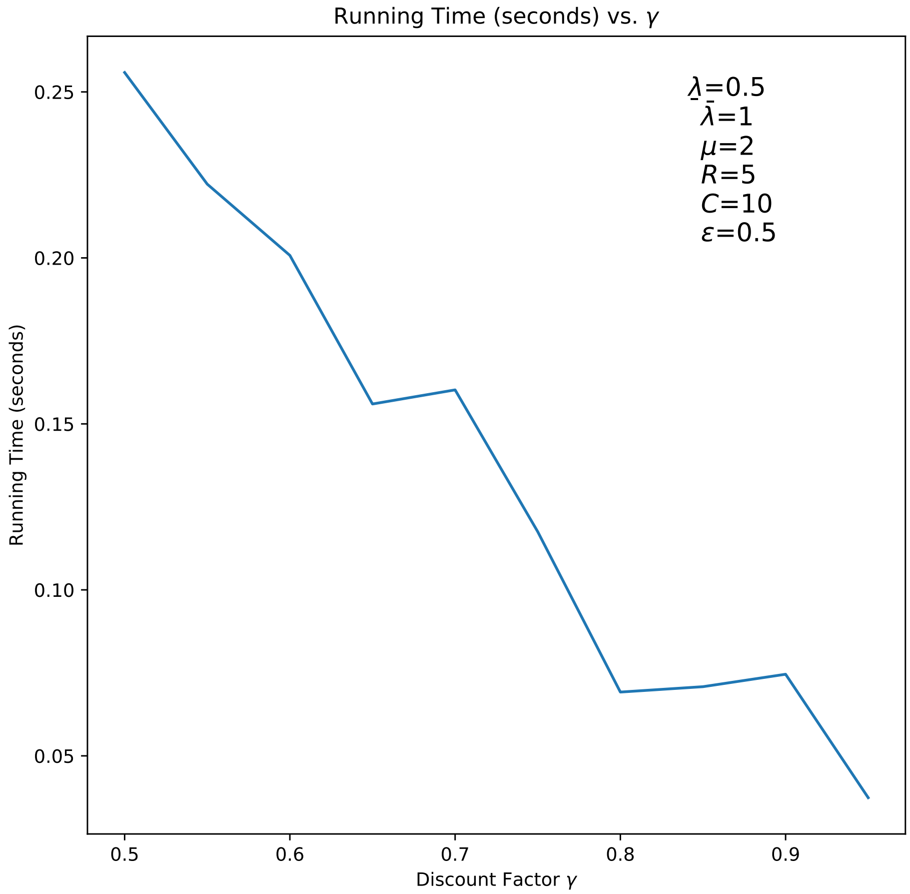

Figure 4.32: Pareto service Uniform cost, Revenue Rate Optimality, case 4, Various Discount Factor, Execution Time

Next we investigate the sensitivity of the optimal social benefit rate and the optimal threshold to various parameters. We use the same parameter settings in Tables 4.1 and 4.2.

According to figures 4.33, 4.37, 4.41, 4.45, the optimal social benefit rate function $S^*(n)$ is non-increasing in n.

According to figures 4.34, 4.38, 4.42, 4.46, the optimal queue length threshold n^* is non-decreasing in the expected service value R and non-increasing in the expected waiting cost rate C.

When the waiting cost rate C is fixed, $S^*(n)$ are non-decreasing in the expected service value R (figures 4.33, 4.41). When the expected service value R is fixed, $S^*(n)$ is non-increasing in the expected waiting cost rate C (figures 4.37), but in the Pareto distribution case, $S^*(n)$ seems to be non-increasing in C when the queue length is below certain value (figure 4.45).

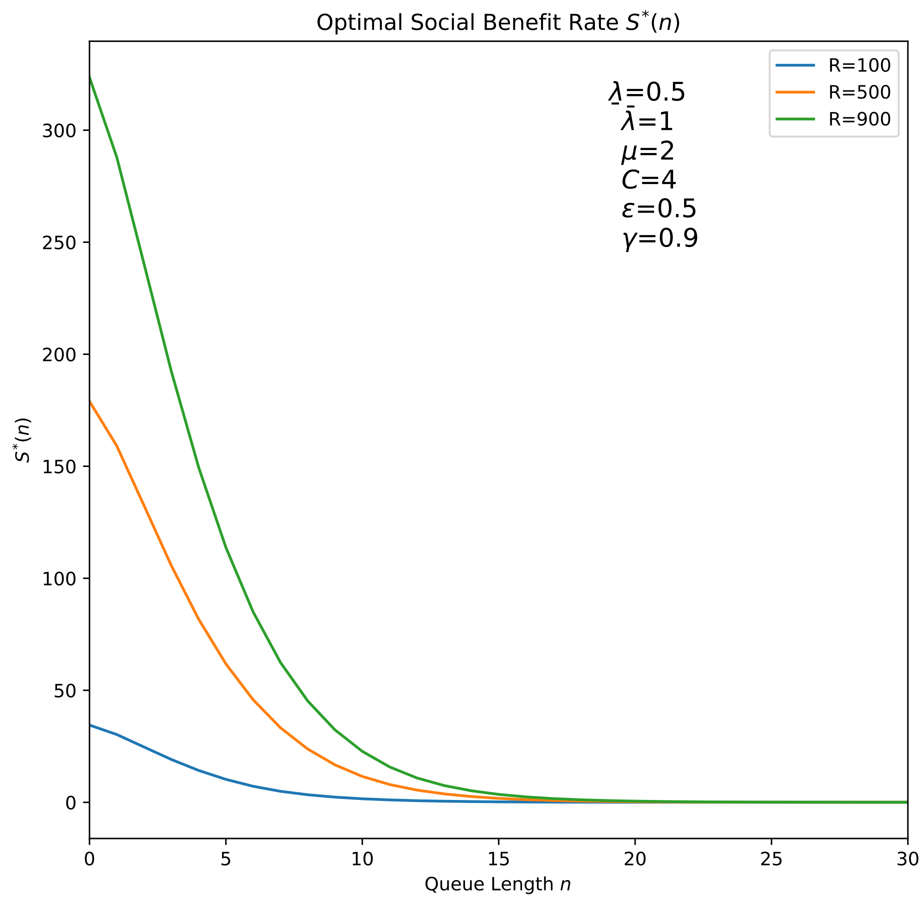

Figure 4.33: Expo service Uniform cost, Social Benefit Rate Optimality, case 1, Various Expected Service Value, Social Benefit Rate

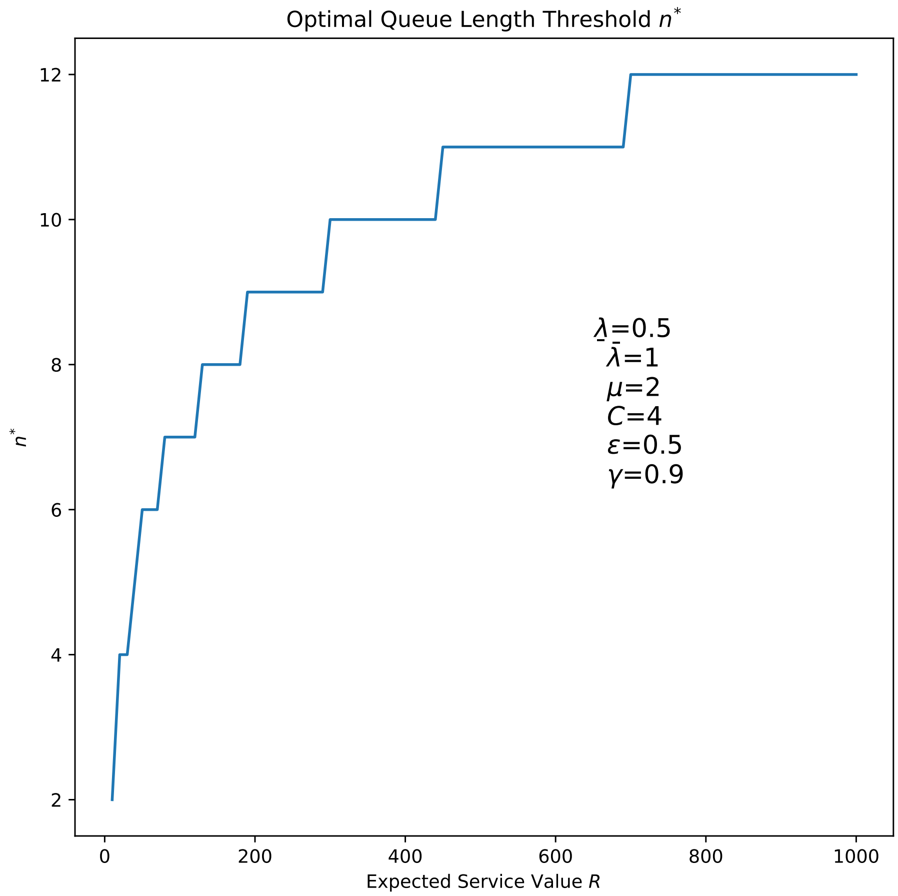

Figure 4.34: Expo service Uniform cost, Social Benefit Rate Optimality, case 1, Various Expected Service Value, Optimal Threshold

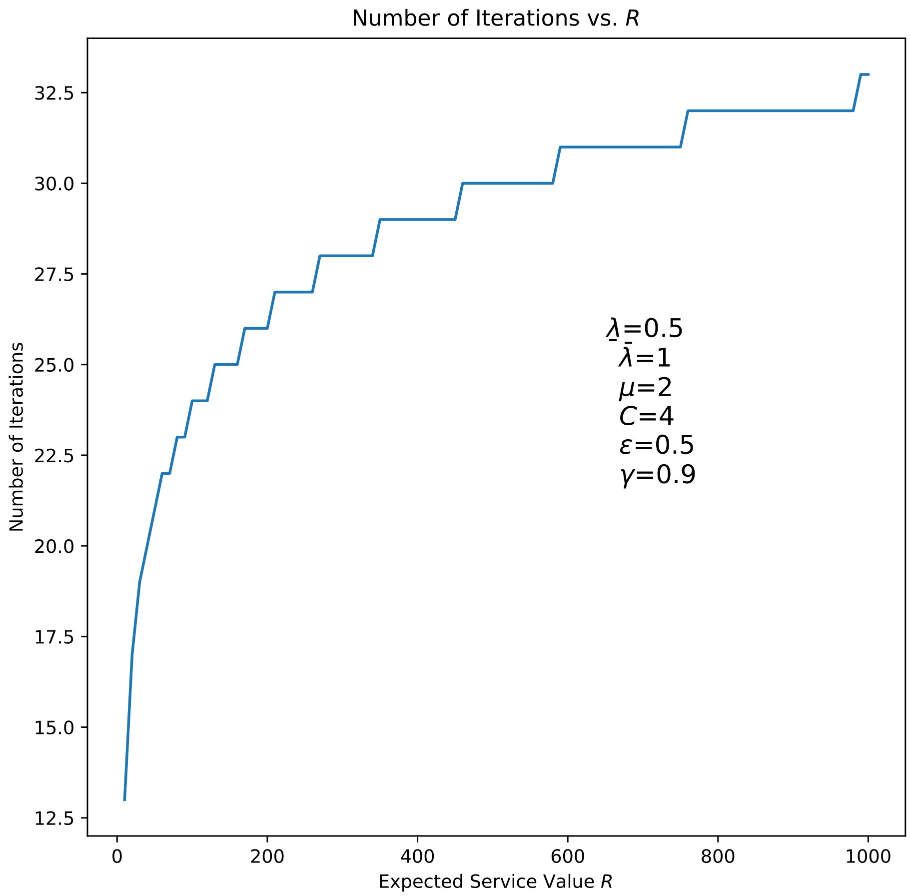

Figure 4.35: Expo service Uniform cost, Social Benefit Rate Optimality, case 1, Various Expected Service Value, Algorithm Iterations

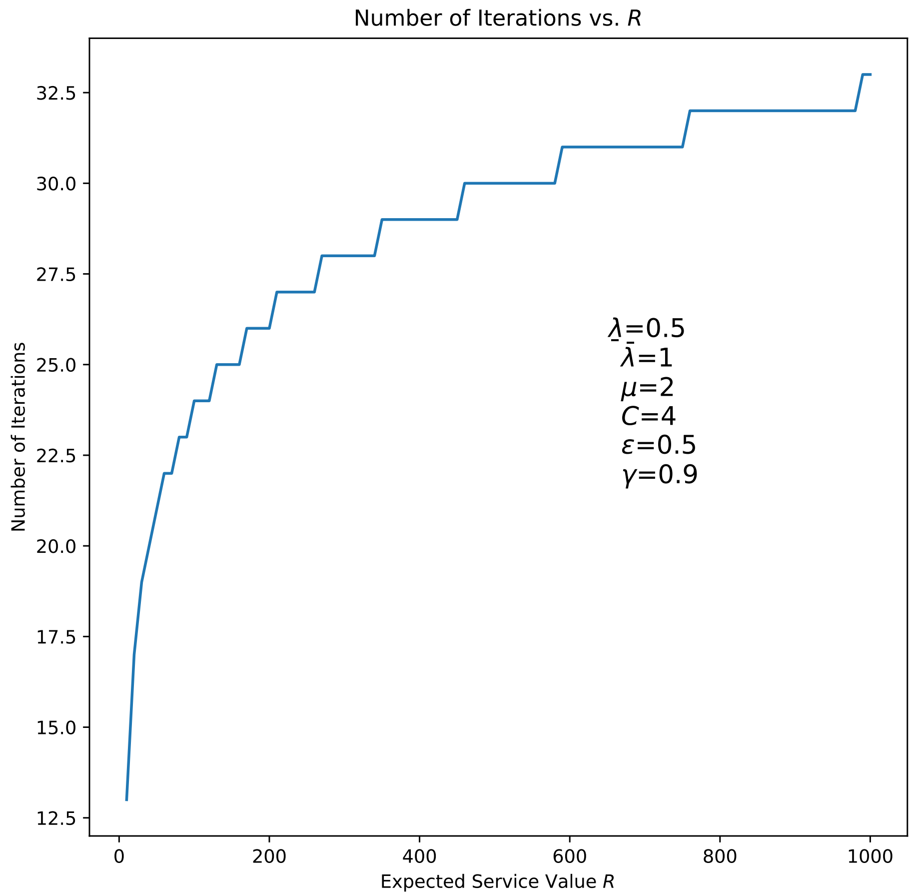

Figure 4.36: Expo service Uniform cost, Social Benefit Rate Optimality, case 1, Various Expected Service Value, Execution Time

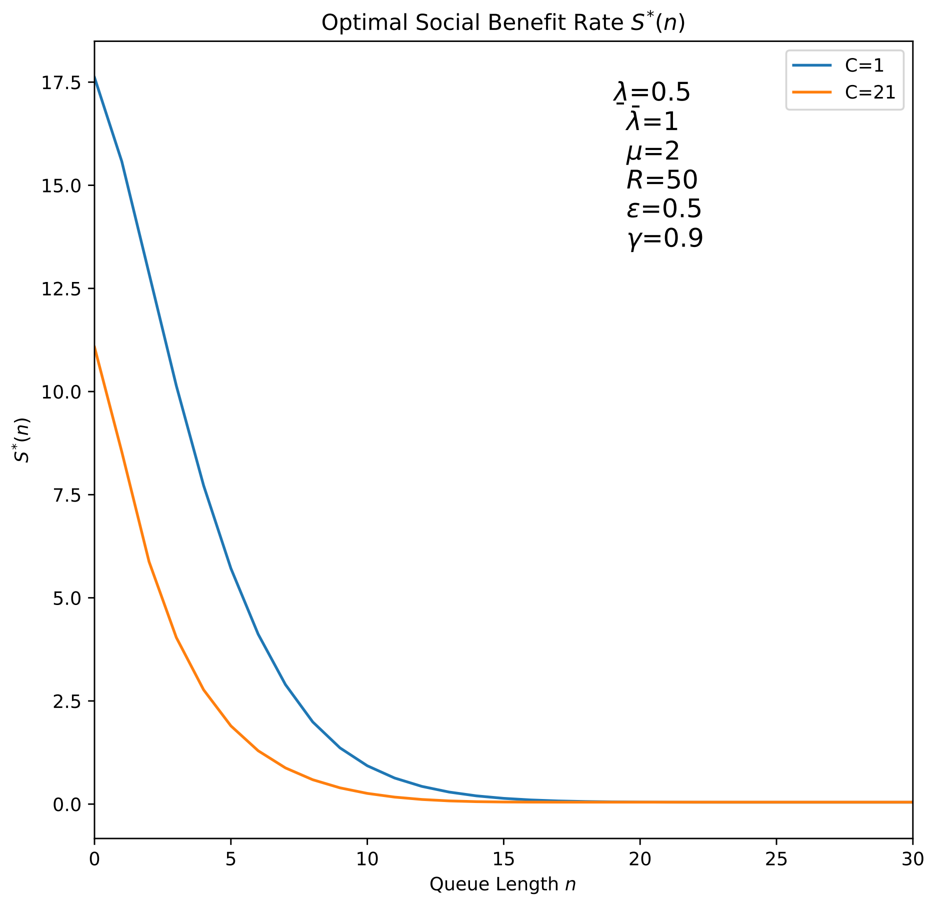

Figure 4.37: Expo service Uniform cost, Social Benefit Rate Optimality, case 2, Various Expected Service Value, Social Benefit Rate

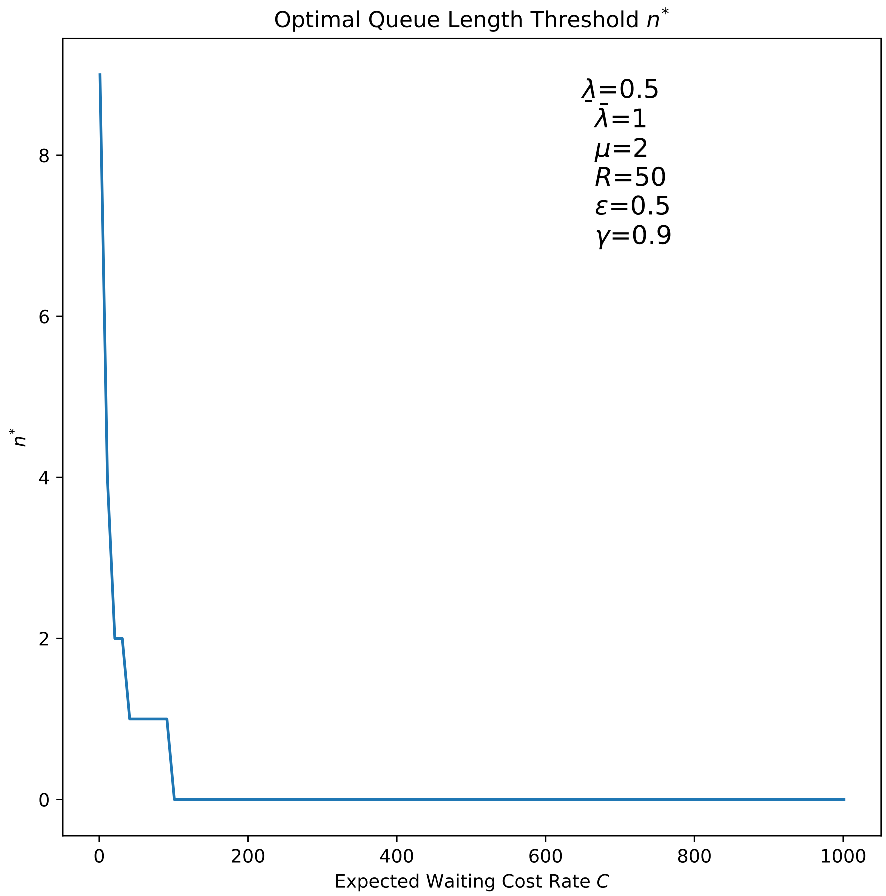

Figure 4.38: Expo service Uniform cost, Social Benefit Rate Optimality, case 2, Various Expected Service Value, Optimal Fee

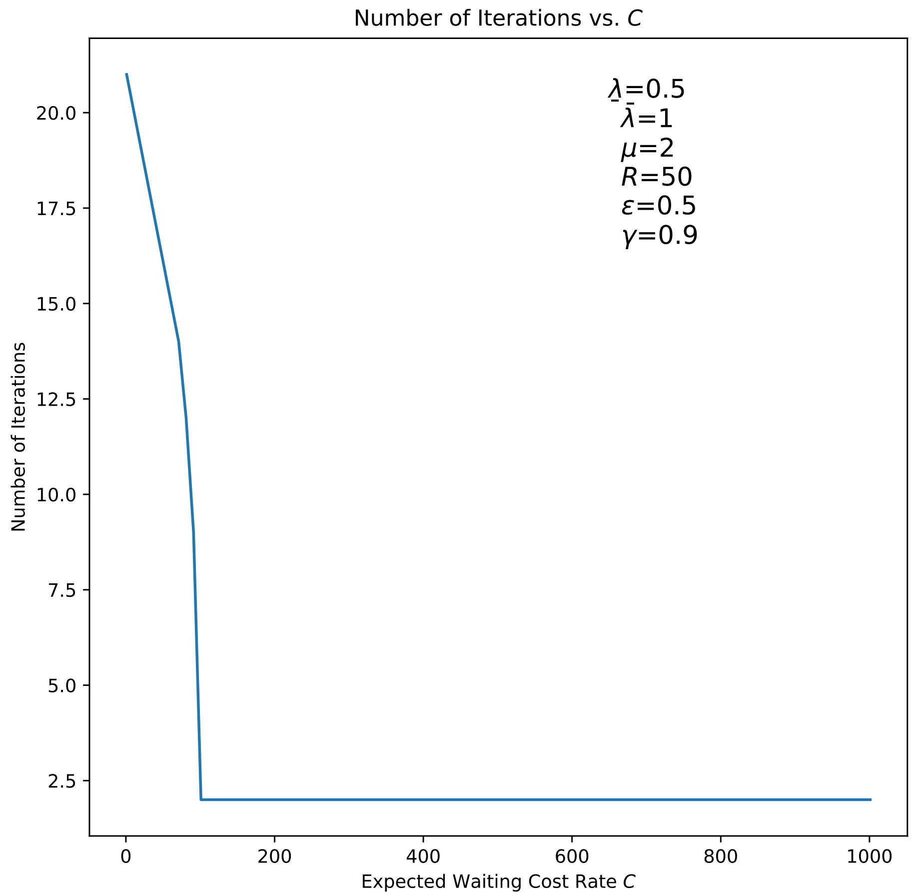

Figure 4.39: Expo service Uniform cost, Social Benefit Rate Optimality, case 2, Various Expected Service Value, Algorithm Iterations

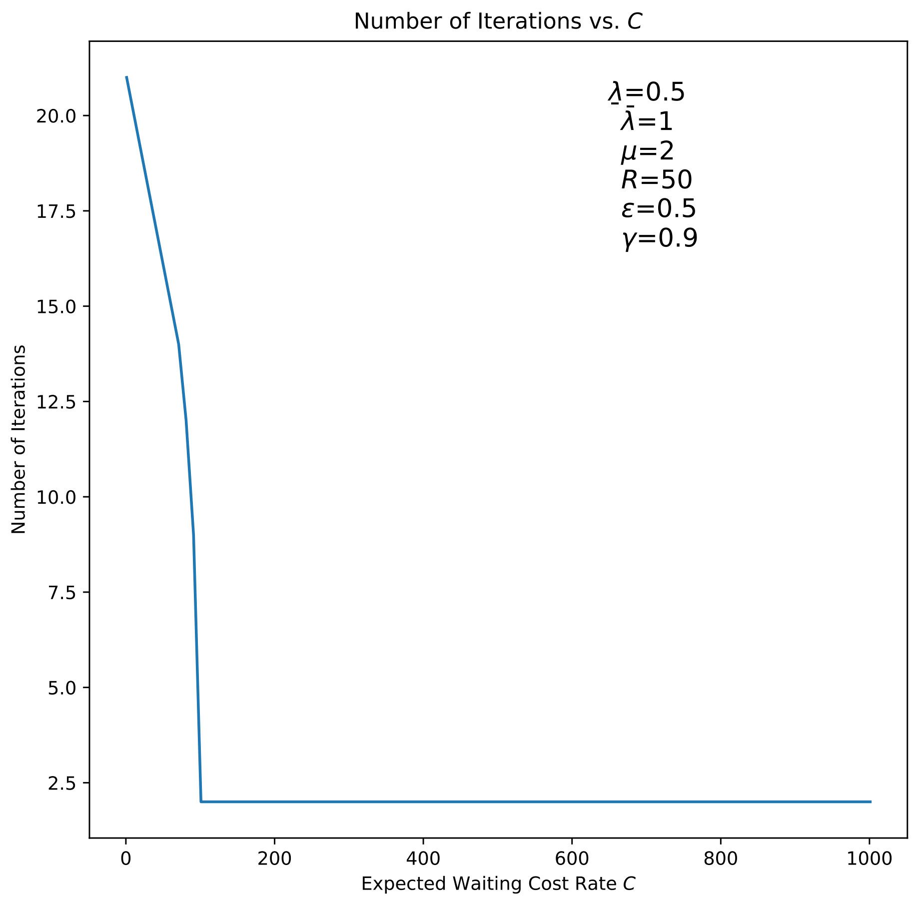

Figure 4.40: Expo service Uniform cost, Social Benefit Rate Optimality, case 2, Various Expected Service Value, Execution Time

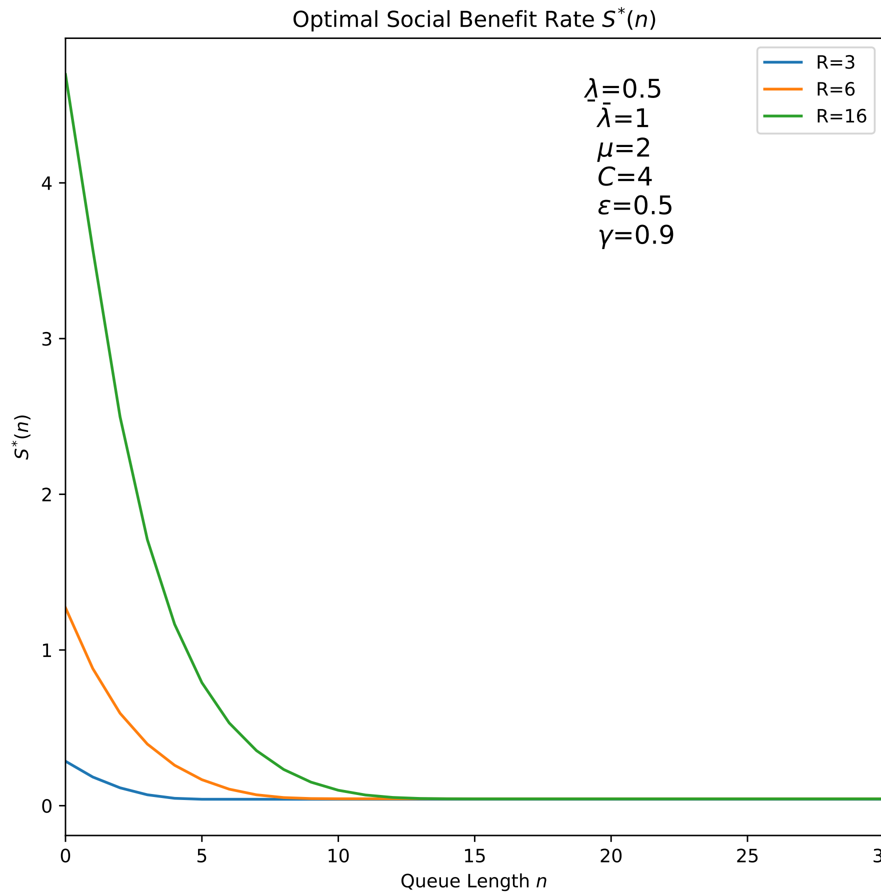

Figure 4.41: Pareto service Uniform cost, Social Benefit Rate Optimality, case 1, Various Expected Service Value, Social Benefit Rate

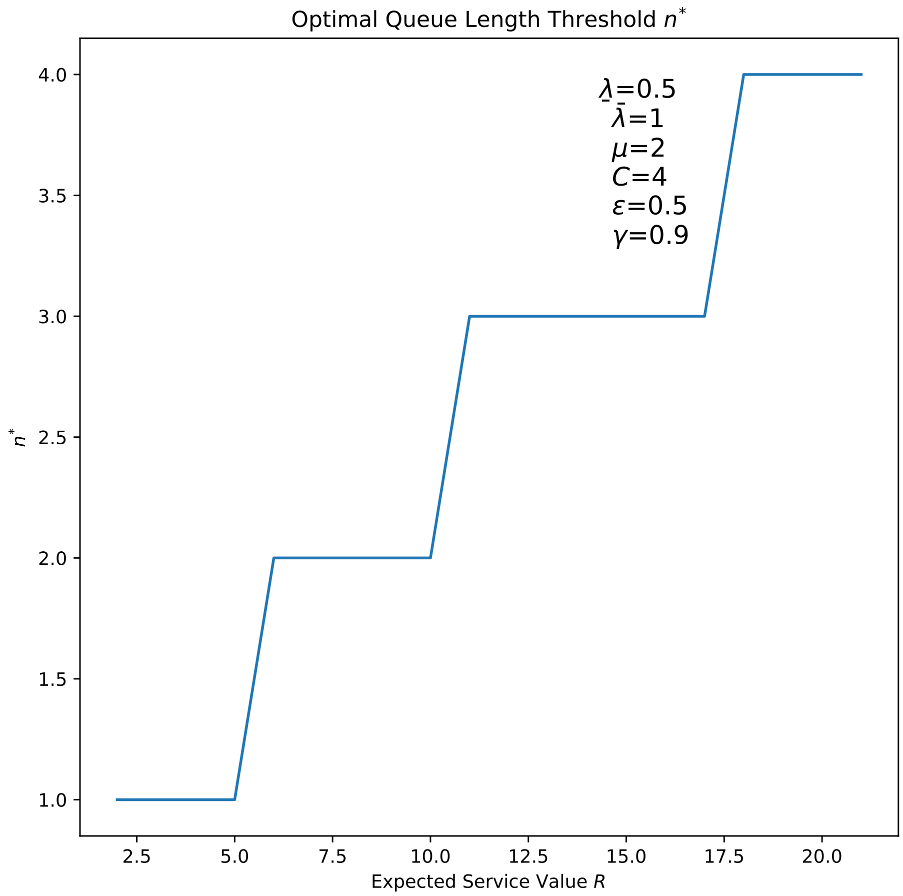

Figure 4.42: Pareto service Uniform cost, Social Benefit Rate Optimality, case 1, Various Expected Service Value, Optimal Threshold

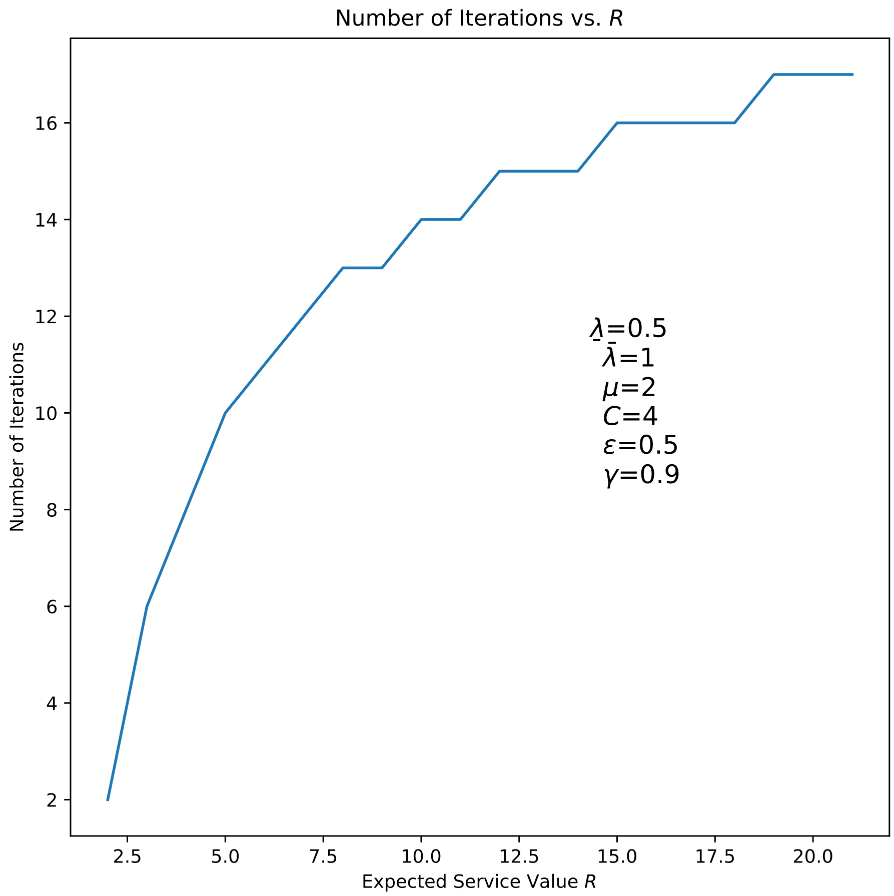

Figure 4.43: Pareto service Uniform cost, Social Benefit Rate Optimality, case 1, Various Expected Service Value, Algorithm Iterations

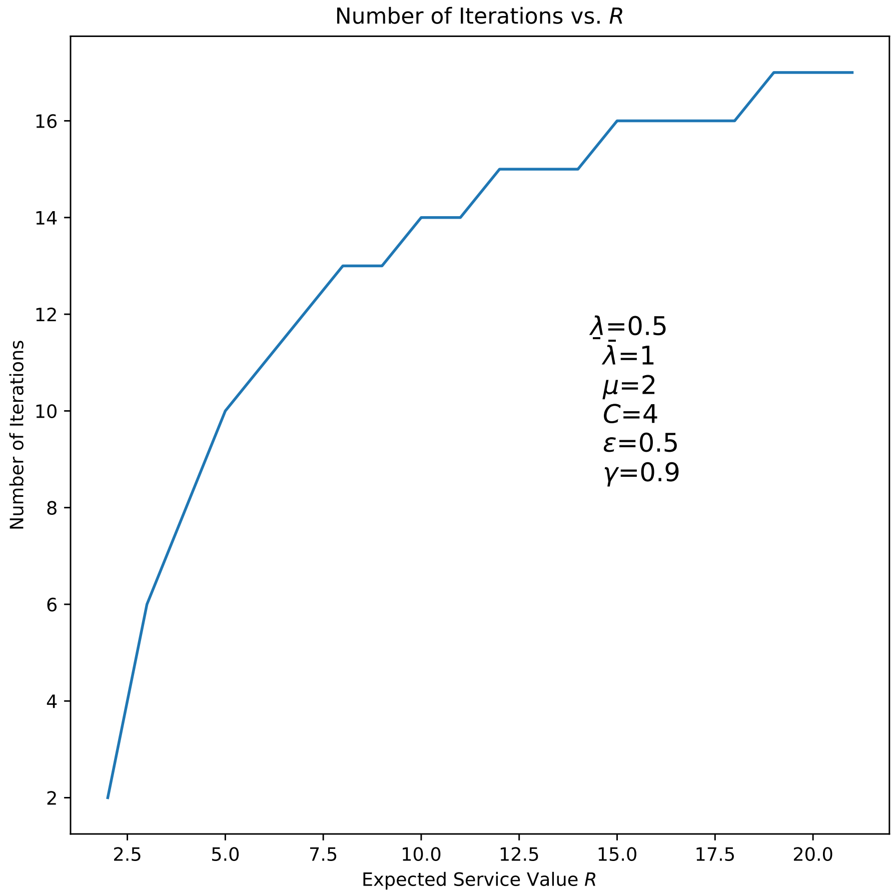

Figure 4.44: Pareto service Uniform cost, Social Benefit Rate Optimality, case 1, Various Expected Service Value, Execution Time

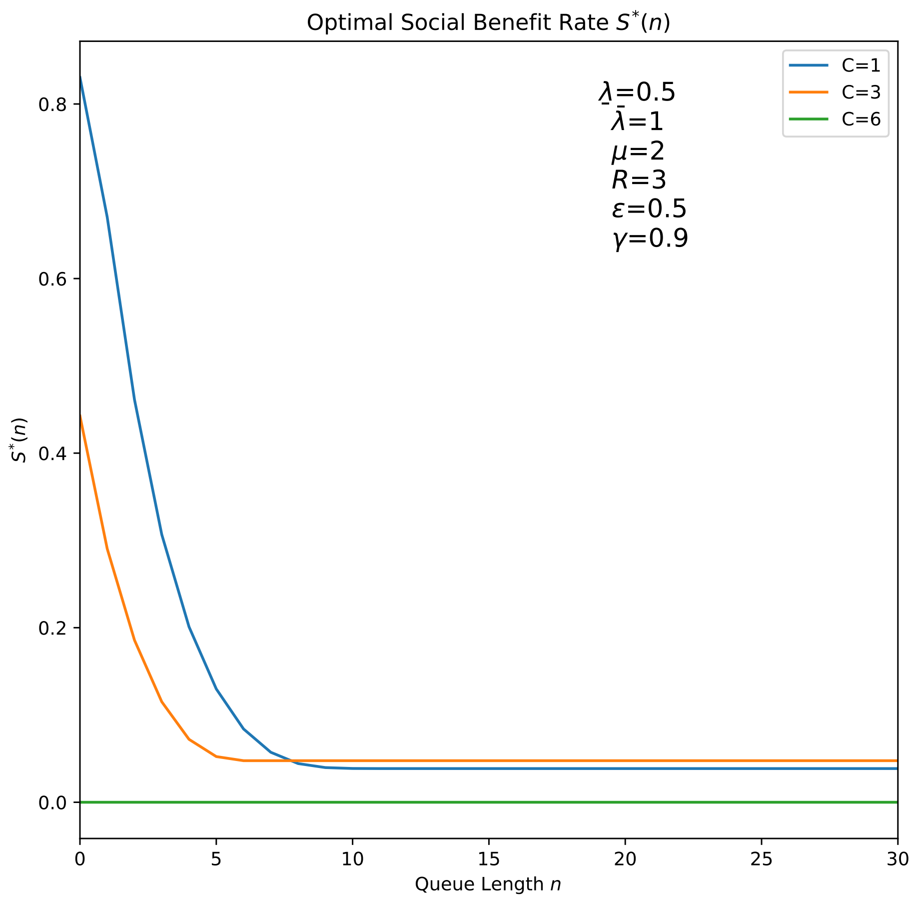

Figure 4.45: Pareto service Uniform cost, Social Benefit Rate Optimality, case 2, Various Expected Service Value, Social Benefit Rate

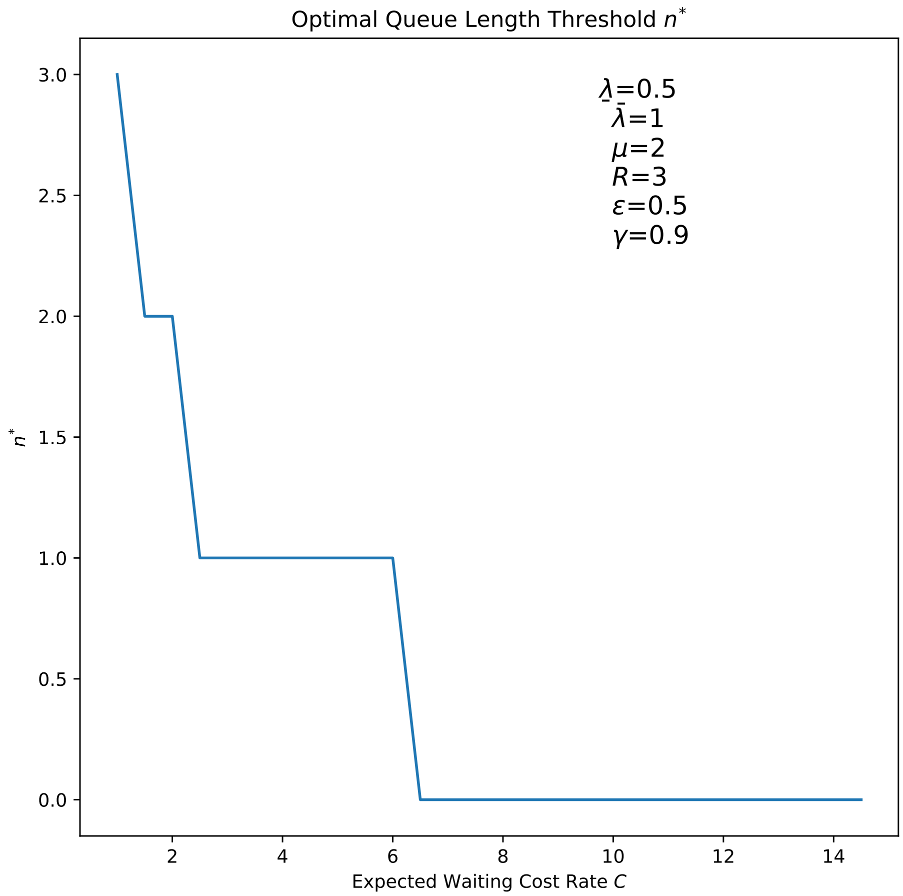

Figure 4.46: Pareto service Uniform cost, Social Benefit Rate Optimality, case 2, Various Expected Service Value, Optimal Fee

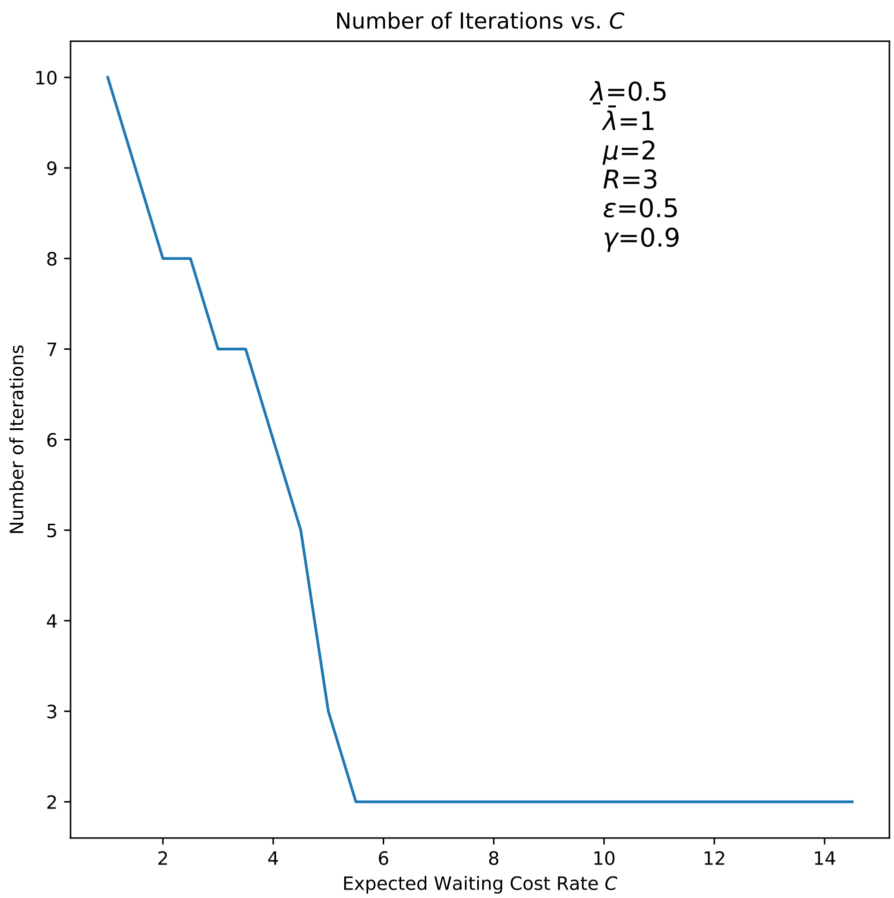

Figure 4.47: Pareto service Uniform cost, Social Benefit Rate Optimality, case 2, Various Expected Service Value, Algorithm Iterations

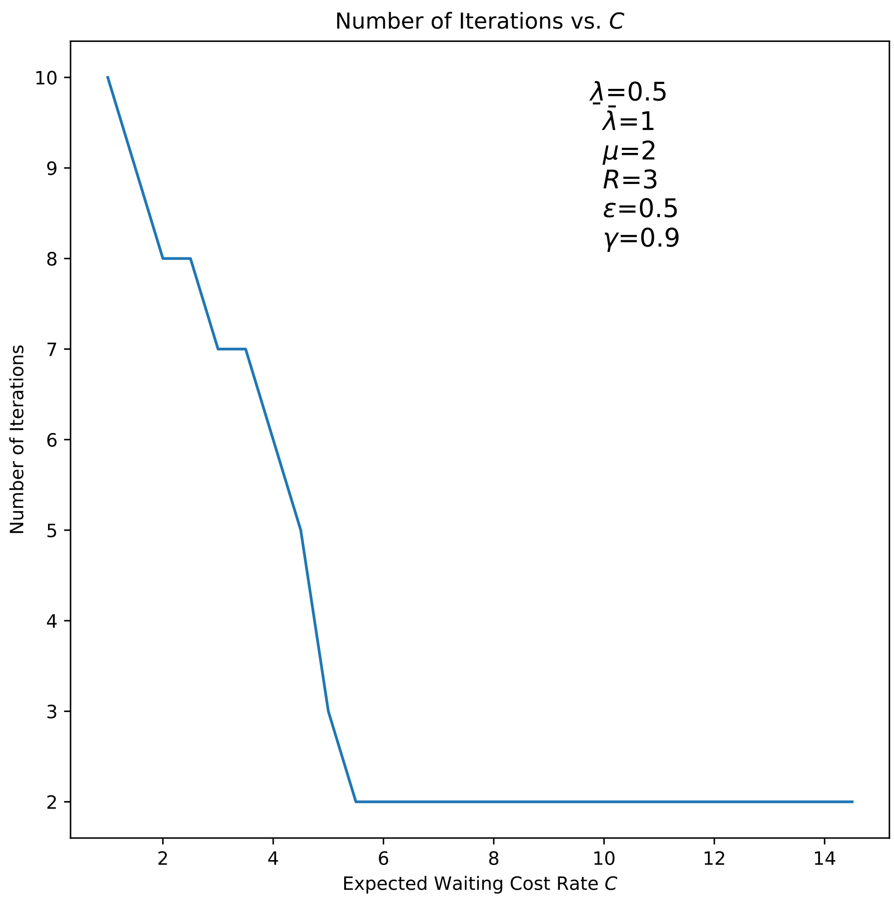

Figure 4.48: Pareto service Uniform cost, Social Benefit Rate Optimality, case 2, Various Expected Service Value, Execution Time

Chapter 5

Conclusion

In this chapter we give a summary of the main results in this research. We also discuss the main contributions of our work.

5.1 Summary

In this research, we extend Naor's observable queueing model into three main models by considering: uncertain arrival rate, heterogeneous customers, static and state-dependent pricing policies.

In Chapter 2, we construct a model in which the arrival rate is uncertain and customers have different economic characteristics. We assume the arrival rate is a proper non-negative random variable whose value is not observed by either customers or the system manager. But the system manager has the information of the arrival rate distribution. We model the customer heterogeneity via a general joint distribution and prove that such a system is stable with probability 1 in Theorem 2.2.4. We derive the revenue rate and the social benefit rate in Theorem 2.3.1. We give a limiting property of the revenue rate, as the admission fee goes infinity, in Theorem 2.3.2. The property is important to other results on the revenue rate, especially when we consider “unimodality”, which holds in the Naor’s basic model. The social benefit rate in general has a complicated formula, as indicated in Theorem 2.3.1. However, it can be simplified when customers have exponentially distributed service values and identical waiting cost rates. We show that $p_S^* \leq p_Z^*$, as long as $S(p)$ and $Z(p)$ are unimodal and continuously differetiable in Theorem 2.4.2. Finally, we examine two special examples in which either the service value is discrete, or the waiting cost rate is uniformly distributed. We show computational results which indicate the revenue rate and the social benefit rate have multiple modes in terms of the admission fee.

In Chapter 3, we consider a model in which the arrival rate is uncertain and customers have identical economic characteristics, but the admission fee is allow to change as the queue length changes. We assume natures chooses the arrival rate value from a given set. The original problem of the SO or RM can be modeled by a CTMDP, in which the queue length is the system state. The state space is discrete and finite. The action space is continuous and closed. We construct a dynamic adversarial model, in which nature may choose different values for the arrival rate as the queue length changes. We convert the CTMDP into a DTMDP and establish Bellman’s equations for the RM’s problem based on a max-min criterion. We give the properties of the optimal value function of the Bellman equations in Proposition 3.3.2. It is important that when the customers are homogeneous, there is a maximal fee that attracts new customers to enter the system at each queue length. We show that in the RM’s problem, customers follow an optimal threshold policy. The optimal policy for nature is to choose the lower bound of the random arrival rate. We prove that the SO and the RM have the same optimal threshold, which is similar to the results in Chen and Frank [5]. Finally, we provide computational examples in which we study how the optimal threshold is affected by the lower and upper bounds of the random arrival rate.

In Chapter 4, we extend the model of Chapter 3 by allowing the customer characteristics to be random. As in Chapter 3, we model the RM's problem by a DTMDP and derive Bellman's equations. We prove that the mapping defined in the Bellman equations is a contraction mapping in Lemma 4.2.1. However, when the customers are heterogeneous, the optimal pricing policy is in general not an explicit function of the queue length. In Section 4.4, we construct an algorithm to solve for the optimal pricing policy. We assume the service value and the waiting cost rate are independent and consider two types of distribution of the random service value. The waiting cost rate is modeled as a uniformly distributed random variable. In theorem 4.4.2 we show that the algorithm can be simplified and the optimal pricing policy is obtained as an explicit function of the queue length. We give computational results to assess the sensitivity of the optimal policy to various parameters. Especially, we find that when the service value follows a Pareto distribution, the optimal policy is a non-decreasing function of the queue length. For the SO's problem, the Bellman equations imply that the optimal policy is a threshold policy. This is because the SO cannot observe individual values, so the customers are either all admitted into the system or all rejected.

5.2 Contributions

In most existing models, the system manager (SO or RM) has complete information of the arrival rate. The optimal pricing policy also depends on the value of the arrival rate. However, we are motivated by the fact that in reality the arrival rate often needs to be estimated. Furthermore, because of the uncertainty of the arrival rate, the stability of the system is of interest. We find limited research on this topic. In Chapter 2, we discuss the conditions of system stability and prove that the system is stable almost surely under mild conditions. This conclusion is important for further study of the system optimality. It also can be extended to other models with various settings. In this chapter, we also observe some examples of interest. In those cases, the revenue rate and the social benefit rate are not unimodal functions of the admission fee. We have seen an example given by Edelson and Hilderbrand [7], in which the waiting cost rate is a two-category variable. However, in the case when customers have exponentially distributed service values and degenerate waiting cost rates, the revenue rate and the social benefit rate seem to be unimodal from the computational results. Unfortunately the proof is not easy, so we assume unimodality and show that $p_S^* \leq p_Z^*$ in Theorem 2.4.2.

Chen and Frank [5] investigate a state-dependent model with homogeneous customers. We extend their model in Chapters 3 and 4. We introduce a robust MDP framework and study a dynamic adversarial problem in which we allow nature to choose different values for the arrival rate as the queue length changes. The problem is formulated in a game-theoretic manner. In

Chapter 3, we use a max-min criterion and conclude that the RM and the SO have the same optimal threshold, and nature chooses the lower bound of the random arrival rate. Future work could involve examining similar systems with different optimality criteria.

In Chapter 4 we investigate a more complicated model than the model of Chapter 3, by considering heterogeneous customers. We first formulate the robust Bellman equations for the RM's problem. We prove that the Bellman equations characterize the optimal value function. This is fundamental to the algorithm we later introduce in this chapter. In order to solve for the optimal pricing policy, we use value iteration algorithm with error bounds [3]. In general a numerical search algorithm is used for solving the maximization problem defined in the Bellman equations. We derive the optimal pricing policy based on unimodality property in a case when customers have exponential service values and uniform waiting cost rates. The optimal fee is an explicit function of the queue length. In addition, we investigate the impact of heavy-tailed distribution of customer heterogeneity. We give an example in which the service value follows a Pareto distribution and the waiting cost rate follows a uniform distribution. The optimal pricing policy seems to be a non-decreasing function of the queue length from the numerical results. This also indicates more future work on the topic of heavy-tailed distributions, which is an interesting extension of this research. Last, we prove that the optimal pricing policy for the SO is of the threshold type, which is similar to Chen and Frank's results [5]. They prove that there is no different in terms of the optimal threshold when customers are homogeneous or heterogeneous. However, in our model, the optimal threshold is different than the case of homogeneous customers.

5.3 Future Work

Our work is based on Naor's $M/M/1$ observable model. There are several possible extensions for future work. One of the assumptions in our models is that customers are not allowed to renege. However, when customers are heterogeneous and may renege, the problem could be challenging but more applicable to real life scenarios. Another interesting extension is to consider a multi-server model with arrival rate uncertainty. In such case each server could have different belief on the arrival rate since customers could choose from different queues. If each queue applies state-dependent pricing, the optimal behavior of customers is affected by multiple queue lengths and fees. Last but not least, we believe investigating a heavy-tailed customer heterogeneity is also of interest, since the optimal policy relies much on the randomness of customer characteristics.

Appendix

Appendix 1

Appendix for Static Pricing with Uncertain Arrival Rate and Heterogeneous Customers

We use a modification of the lemma below to prove Theorem 2.4.2. We provide a more general statement here, as it may of use in subsequent analysis.

Lemma 1.0.1. *For an $M^\Lambda/M/1$ system with heterogeneous customers we have:*

1. *$\pi_0(p)$ is a non-decreasing function of p.*

2. *$\lim_{p\to\infty} \pi_0(p) = 1$.*

Proof. First, by (2.6), we have that for all $p \geq 0$

$$\pi_0(p) = E_\Lambda\left(\frac{1}{M}\right) = \int_0^\infty \frac{1}{1+\sum_{n=1}^\infty \prod_{i=1}^n \left(\frac{\lambda}{\mu}\bar{F}_{\Theta_i}(p)\right)} f_\Lambda(\lambda)\mathrm{d}\lambda. \tag{1.1}$$

We note that $\pi_0(p)$ is positive for all $p \geq 0$ and almost all realizations of Λ by Theorem 2.2.4. Since $\bar{F}_{\Theta_i}(p)$ is a ccdf, it is non-increasing for all i. Then, it is clear from (1.1) that π_0 is non-decreasing in p.

Next, we prove the limit result. By Lemma 2.2.3, $\forall\lambda \geq 0$ and all finite n positive integers n

$$\lim_{p\to\infty} \prod_{i=1}^n \left(\frac{\lambda}{\mu}\bar{F}_{\Theta_i}(p)\right) = 0.$$

We again recall that $\bar{F}_{\Theta_i}(p)$ is non-increasing in p and thus

$$\prod_{i=1}^{n}\left(\frac{\lambda}{\mu}\bar{F}_{\Theta_i}(p)\right) \leq \prod_{i=1}^{n}\left(\frac{\lambda}{\mu}\bar{F}_{\Theta_i}(0)\right)$$

for all $p \geq 0$, all $\lambda \geq 0$ and positive n. Now, using the stability result (Theorem 2.2.4), we have $\sum_{n=1}^{\infty}\prod_{i=1}^{n}\left(\frac{\lambda}{\mu}\bar{F}_{\Theta_i}(0)\right) < \infty$. Hence, for each $p \geq 0$

$$\prod_{i=1}^{n}\left(\frac{\lambda}{\mu}\bar{F}_{\Theta_i}(p)\right)$$

is dominated by a summable function. Applying dominated convergence, we obtain

$$\lim_{p\to\infty}\sum_{n=1}^{\infty}\prod_{i=1}^{n}\left(\frac{\lambda}{\mu}\bar{F}_{\Theta_i}(p)\right) = \sum_{n=1}^{\infty}\lim_{p\to\infty}\prod_{i=1}^{n}\left(\frac{\lambda}{\mu}\bar{F}_{\Theta_i}(p)\right) = 0.$$

Next, since $\bar{F}_{\Theta_i}(p)$ is always non-negative we have

$$0 \leq \frac{1}{1+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\left(\frac{\lambda}{\mu}\bar{F}_{\Theta_i}(p)\right)} \leq 1.$$

Hence this expression is uniformly dominated by 1, which is integrable against $f_\Lambda(\lambda)$. Now, returning to (1.1), again applying dominated convergence and basic properties of limits, we finally conclude that

$$\begin{aligned}\lim_{p\to\infty}\pi_0(p) &= \int_0^{\infty}\lim_{p\to\infty}\frac{1}{1+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\left(\frac{\lambda}{\mu}\bar{F}_{\Theta_i}(p)\right)}f_\Lambda(\lambda)\mathrm{d}\lambda \\ &= \int_0^{\infty}\frac{1}{1+\lim_{p\to\infty}\sum_{n=1}^{\infty}\prod_{i=1}^{n}\left(\frac{\lambda}{\mu}\bar{F}_{\Theta_i}(p)\right)}f_\Lambda(\lambda)\mathrm{d}\lambda = 1\end{aligned}$$

□

Proof of Theorem 2.3.2. First we note

$$Z = \mu p\left(1-\pi_0(p)\right) = \mu\int_0^{\infty}\frac{\sum_{n=1}^{\infty}\frac{\lambda^n}{\mu^n}p\prod_{i=1}^{n}\bar{F}_{\Theta_i}(p)}{1+\sum_{n=1}^{\infty}\frac{\lambda^n}{\mu^n}\prod_{i=1}^{n}\bar{F}_{\Theta_i}(p)}f_\Lambda(\lambda)\mathrm{d}\lambda.$$

If $\mathbb{E}[\Theta_n^2] < \infty \ \forall n \geq 1$, then

$$\lim_{p\to\infty} p\bar{F}_{\Theta_n}(p) = 0,$$

by observing that the bound on the second moment implies

$$\int_0^\infty p\bar{F}_{\Theta_n}(p) < \infty.$$

Further, we have $\forall n \geq 1$, $\lim_{p\to\infty} \bar{F}_{\Theta_i}(p) = 0$. Hence, $\forall n \geq 1$,

$$\lim_{p\to\infty} p \prod_{i=1}^{n} \bar{F}_{\Theta_i}(p) = 0.$$

Thus for all $\lambda \geq 0$ and $\mu > 0$,

$$\lim_{p\to\infty} \frac{\lambda^n}{\mu^n} p \prod_{i=1}^{n} \bar{F}_{\Theta_i}(p) = 0.$$

Next, since $\lim_{p\to\infty} p\bar{F}_{\Theta_1}(p) = 0$, the function has a finite maximum on $[0, \infty)$. In other words, for some finite K, $p\bar{F}_{\Theta_1}(p) \leq K$, for all $p \geq 0$. Then we have that

$$\frac{\lambda^n}{\mu^n} p \prod_{i=1}^{n} \bar{F}_{\Theta_i}(p) \leq \frac{\lambda^n}{\mu^n} K,$$

for all $p \geq 0$. Further, the dominating function on the right-hand side is summable, as

$$\sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n} K = \frac{K\rho}{1-\rho} < \infty,$$

where we set $\rho = \lambda/\mu$. Notice that $\rho < 1$ for all λ, by our assumption in the theorem statement. Then, applying dominated convergence we have

$$\lim_{p\to\infty} \sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n} p \prod_{i=1}^{n} \bar{F}_{\Theta_i}(p) = \sum_{n=1}^{\infty} \lim_{p\to\infty} \frac{\lambda^n}{\mu^n} p \prod_{i=1}^{n} \bar{F}_{\Theta_i}(p) = 0,$$

This further implies that

$$\lim_{p\to\infty} \frac{\sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n} p \prod_{i=1}^{n} \bar{F}_{\Theta_i}(p)}{1+\sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n} \prod_{i=1}^{n} \bar{F}_{\Theta_i}(p)} = 0.$$

Furthermore, since the denominator is always greater than or equal to 1, we have

$$\frac{\sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n} p \prod_{i=1}^{n} \bar{F}_{\Theta_i}(p)}{1+\sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n} \prod_{i=1}^{n} \bar{F}_{\Theta_i}(p)} \leq \sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n} p \prod_{i=1}^{n} \bar{F}_{\Theta_i}(p) \leq \frac{K\rho}{1-\rho},$$

as argued above. The dominating function on the right-hand side is integrable against $f_\Lambda(\lambda)$, by virtue of the bound in the theorem statement. Hence, with yet another application of dominated convergence, we obtain the final result:

$$\begin{aligned}
\lim_{p\to\infty} Z(p) &= \mu \lim_{p\to\infty} \int_0^{\infty} \frac{\sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n} p \prod_{i=1}^{n} \bar{F}_{\Theta_i}(p)}{1+\sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n} \prod_{i=1}^{n} \bar{F}_{\Theta_i}(p)} f_\Lambda(\lambda) \mathrm{d}\lambda \\
&= \mu \int_0^{\infty} \lim_{p\to\infty} \frac{\sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n} p \prod_{i=1}^{n} \bar{F}_{\Theta_i}(p)}{1+\sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n} \prod_{i=1}^{n} \bar{F}_{\Theta_i}(p)} f_\Lambda(\lambda) \mathrm{d}\lambda = 0.
\end{aligned}$$

□

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Vita

Chengcheng Liu was born in Suihua, China. She joined the Operations Research and Industrial Engineering program at The University of Texas at Austin as a doctoral student in Fall 2013. Prior to that, She received a Master's degree in Statistics from Columbia University in the City of New York in Spring 2013. She finished her Bachelor's degree in Statistics from the Renmin University of China, Beijing, China in 2011. She completed research with Dr. John J. Hasenbein on Stability and Pricing in Naor's Model with Arrival Rate Uncertainty. She is currently working as a Data Scientist in IBM located in Foster City, California.

Address: cl35622@utexas.edu

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